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I.—PROPOSITIONS IN THE THEORY OF ATTRACTION.

PART II.

I SHALL now prove a general theorem, which comprehends the propositions demonstrated in Part I., along with several others of importance in the theories of electricity and heat.

Let M and M_1 be two bodies, or groups of attracting or repelling points; and let v and v_1 be their potentials on xyz; let R and R_1 be their total attractions on the same point; and let θ be the angle between the directions of R and R_1, and $\alpha\beta\gamma$, $\alpha_1\beta_1\gamma_1$, the angles which they make with xyz. Let S be a closed surface, ds an element, corresponding to the co-ordinates xyz; and P and P_1 the components of R, R_1, in a direction perpendicular to the surface at ds. Then we have

$$R\cos\alpha = -\frac{dv}{dx}, \quad R\cos\beta = -\frac{dv}{dy}, \quad R\cos\gamma = -\frac{dv}{dz},$$

$$R_1\cos\alpha_1 = -\frac{dv_1}{dx}, \quad R_1\cos\beta_1 = -\frac{dv_1}{dy}, \quad R_1\cos\gamma_1 = -\frac{dv_1}{dz},$$

$$\cos\theta = \cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1;$$

hence, $$\frac{dv}{dx}\frac{dv_1}{dx} + \frac{dv}{dy}\frac{dv_1}{dy} + \frac{dv}{dz}\frac{dv_1}{dz} = RR_1\cos\theta.$$

Hence, $\iiint RR_1\cos\theta\, dx\,dy\,dz$

$$= \iiint\left(\frac{dv}{dx}\frac{dv_1}{dx} + \frac{dv}{dy}\frac{dv_1}{dy} + \frac{dv}{dz}\frac{dv_1}{dz}\right)dx\,dy\,dz \;\ldots\ldots (a),$$

where we shall suppose the integrals to include every point in the interior of S. Now, by integration by parts, the second member may be put under the form,

$$\iint v_1\left(\frac{dv}{dx}\,dy\,dz+\frac{dv}{dy}\,dx\,dz+\frac{dv}{dz}\,dx\,dy\right)$$
$$-\iiint v_1\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)dx\,dy\,dz\ldots(b);$$

where the double integrals are extended over the surface S, and the triple integrals as before, over every point in its interior. If we transform the first term of this by (b), Part I., and observe that $-\frac{dv}{dn}=P$, it becomes

$$-\iint v_1\,Pds.$$

Again, $$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=0\ldots\ldots(c).$$

except when xyz is a point of the attracting mass.

If this be the case, and if k be the density of the matter at the point, we have

$$\left.\begin{aligned}\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+4\pi k=0\\ \text{therefore}\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)dx\,dy\,dz+4\pi dm=0\end{aligned}\right\}\ldots\ldots(d).$$

Hence (a) is transformed into

$$\iiint RR_1\cos\theta\,dx\,dy\,dz=4\pi\iiint v_1\,dm-\iint v_1\,Pds\ldots(3);$$

similarly, by performing the integration in (a), on the terms

$$\frac{dv}{dx},\ \frac{dv}{dy},\ \frac{dv}{dz},\ \text{instead of}\ \frac{dv_1}{dx},\ \frac{dv_1}{dy},\ \frac{dv_1}{dz},$$

we should have found

$$\iiint RR_1\cos\theta\,dx\,dy\,dz=4\pi\iiint v dm_1-\iint vP_1\,ds\ldots(4).$$

If the triple integrals in (a) were extended over all the space without S, or over every point between S, and another surface, S', enclosing it, at an infinite distance, it may be shown, as in part I., that the superior values of the double integrals in (b), corresponding to S', vanish. Hence, the inferior values being those which correspond to S, we have, instead of (3) and (4),

$$\iiint RR_1\cos\theta\,dx\,dy\,dz=4\pi\iiint v_1\,dm+\iint v_1\,Pds\ldots\ldots(5),$$
$$\iiint RR_1\cos\theta\,dx\,dy\,dz=4\pi\iiint v\,dm_1+\iint v\,P_1ds\ldots\ldots(6).$$

It is obvious that v and v_1 in these equations may be any functions, each of which satisfy equations (c) and (d), whether we consider them as potentials or temperatures, or as mere analytical functions with the restriction that, in (5) and (6), v

and v_1 must be such as to make $\iint v_1 P ds$ and $\iint v P_1 ds$ vanish at S'. If each of them satisfy (c) for all the points within the limits of the triple integrals considered, dm and dm_1 will each vanish; but if there be any points within the limits, for which either v or v_1 does not satisfy (c), the value of dm or dm_1 at those points will be found from (d).

Thus let $v_1 = 1$, for every point. Then we must have $dm_1 = 0$. Also $R_1 = 0$, $P_1 = 0$.

Hence, (3) becomes

$$\iint P ds = 4\pi \iiint dm = 4\pi m \ldots\ldots\ldots\ldots (7),$$

if m be the part of M within S. This expression is independent of the quantity of matter without S, and if $m = 0$, it becomes

$$\iint P ds = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

If M be a group of sources of heat, in a solid body, P will be the flux across a unit of surface, at the point xyz. Hence the total flux of heat across S is equal to the sum of the expenditures from all the sources in the interior; and if there be no sources in the interior, the whole flux is nothing. Both these results, though our physical ideas of heat would readily lead us to anticipate them, are by no means axiomatic when considered analytically. In exactly a similar manner, Poisson* proves that the total flux of heat out of a body during an instant of time is equal to the sum of the diminutions of heat of each particle of the body, during the same time. This follows at once from (7). For, if we suppose there to be no sources of heat within S, but the temperature of interior points to vary with the time, on account of a non-uniform initial distribution of heat, we have

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \frac{dv}{dt}.$$

Hence, by (d), we must use $-\frac{dv}{dt} dx\, dy\, dz$, instead of $4\pi dm$ and therefore (7) becomes

$$\iint P ds = \iiint \frac{-dv}{dt} dx\, dy\, dz.$$

It was the analysis used by Poisson, in the demonstration of this theorem, that suggested the demonstrations given in part I., of propositions (1) and (2).

As another example of the application of the theorem expressed by (3) and (4), let v_1 be the potential of a unit of mass, concentrated at a fixed point, $x'y'z'$. Hence, $M_1 = 1$, and

* See *Théorie de la Chaleur*, p. 177.

$dm_1 = 0$, except when xyz, at which dm_1 is supposed to be situated, coincides with $x'y'z'$; and, if Δ be the distance of xyz from $x'y'z'$, $v_1 = \dfrac{1}{\Delta}$.

Hence, according as $x'y'z'$ is without or within S,

$$\iiint v\,dm_1 = 0, \quad \text{or} \quad \iint v\,dm_1 = v' \iiint dm_1 = v' \quad \ldots\ldots (e),$$

the triple integrals being extended over the space within S. Now let us suppose M to be such, that v has a constant value (v) at S. Then $\iint vP_1 ds = (v)\iint P_1 ds$, which by (7), is $= 0$, or to $4\pi(v)$, according as $x'y'z'$ is without or within S. Hence, by comparing (3) and (4) we have, in the two cases,

$$4\pi \iiint \frac{dm}{\Delta} - \iint \frac{Pds}{\Delta} = 0, \text{ or } \iint \frac{Pds}{\Delta} = 4\pi \iiint \frac{dm}{\Delta} = 4\pi v' \ldots (9),$$

$$\text{and } \quad 4\pi \iiint \frac{dm}{\Delta} - \iint \frac{Pds}{\Delta} = -4\pi(v) + 4\pi v';$$

$$\text{therefore} \qquad \iint \frac{Pds}{\Delta} = 4\pi(v) \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

These are the two propositions (1) and (2) proved in Part I., which are therefore, as we see, particular cases of the general theorem expressed by (3) and (4).*

If $v = v_1$, and if both arise from sources situated without S, (3) becomes

$$\iiint R^2\, dx\, dy\, dz = \iint vPds \quad \ldots\ldots\ldots\ldots (11),$$

a proposition given by Gauss. If v have a constant value (v), over S, we have

$$\iint vPds = (v)\iint Pds = 0, \quad \text{by (8)},$$

$$\text{hence} \qquad \iiint R^2\, dx\, dy\, dz = 0.$$

Therefore $R = 0$, and $v = (v)$, for interior points. Hence, if the potential produced by any number of sources, have the same value over every point of a surface which contains none of them, it will have the same value for every interior point also. If we consider the sources to be spread over S, it follows that $v = (v)$, at the surface is a condition which implies that the attraction on an interior point will be nothing. Hence the

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* It may be here proper to state that these theorems, which were first demonstrated by Gauss, are the subject of a Mémoire by M. Chasles, in the Additions to the *Connaissance des Temps* for 1845, published in June, 1842. In this Mémoire he refers to an announcement of them, without a demonstration, in the *Comptes Rendus des Séances de l'Académie des Sciences*, Feb. 11th, 1839, a date earlier than that of M. Gauss' Mémoire, which was read at the Royal Society of Gottingen, in March 1840.

sole condition for the distribution of electricity over a conducting surface, is that its attraction shall be every where perpendicular to the surface, a proposition which was proved from indirect considerations, relative to heat, in a former paper.*

In exactly a similar manner, if none of the sources be without S, by means of (5) and (7), it may be shown that

$$\iiint R^2\, dx\, dy\, dz = 4\pi M(v) \ldots\ldots\ldots\ldots (12);$$

the triple integrals being extended over all the space without S. Hence a quantity of matter μ can only be distributed in one way on S, so as to make (v) be constant. For if there were two distributions of μ, each making (v) constant, there would be a third, corresponding to their difference, which would also make (v) constant. The whole mass in the third case would be nothing. Hence, by (12), we must have $\iiint R^2\, dx\, dy\, dz = 0$, and therefore $R = 0$, for external points; and, since (v) is constant at the surface, R must be $= 0$, for interior points also. Now this cannot be the case unless the density at each point of the surface be nothing, on account of the theorem of Laplace, that, if ρ be the density at any point of a stratum which exerts no attraction on interior points, its attraction on an interior point, close to the surface, will be $4\pi\rho$. This important theorem, which shows that there is only one distribution of electricity on a body, that satisfies the condition of equilibrium, was first given by Gauss. It may be readily extended, as has been done by Liouville,† to the case of any number of electrified bodies, influencing one another, by supposing S to consist of a number of isolated portions, which will obviously not affect the truth of (5) and (6).

Then, if we suppose v to have the constant values, (v), $(v)'$, &c., at the different surfaces, and the quantities of matter on these surfaces to be M, M', &c. we should have, instead of (11),

$$\iiint R^2\, dx\, dy\, dz = 4\pi \left\{M(v) + M'(v)' + \&c.\right\} \ldots\ldots (13),$$

and from this it may shown as above, that there is only one distribution of the same quantities of matter, M, M', &c. which satisfies the conditions of equilibrium.

If both M and M_1 be wholly within S, by comparing (5) and (6), or if both be without S, by comparing (3) and (4), we have

$$\iint P v_1 ds = \iint P_1 v ds \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Now let S be a sphere, and let $r\theta\phi$ be the polar co-ordinates, from the centre as pole, of any point in the surface, to

* See Vol. III., p. 74.

† See Note to M. Chasles' Memoire in the *Connaissance des Temps*, for 1845.

which the potentials v and v_1 correspond. Then we shall have $P = -\frac{dv}{dr}$, $P_1 = -\frac{dv_1}{dr}$, and we may assume $ds = r^2 \sin\theta\, d\theta\, d\phi$. Hence (13) becomes

$$\int_0^\pi\int_0^{2\pi} v_1 \frac{dv}{dr} \sin\theta\, d\theta\, d\phi = \int_0^\pi\int_0^{2\pi} v \frac{dv_1}{dr} \sin\theta\, d\theta\, d\phi \ldots\ldots (15).$$

This equation leads at once to the fundamental property of Laplace's coefficients. For if v and v_1 be of the forms $Y_m r^m$, ${}_1Y_n r^n$, m and n being any positive or negative integers, zero included, and Y_m and ${}_1Y_n$ being independent of r, we have, by substitution in (15),

$$m\int_0^\pi\int_0^{2\pi} Y_m\, {}_1Y_n \sin\theta\, d\theta\, d\phi = n\int_0^\pi\int_0^{2\pi} Y_m\, {}_1Y_n \sin\theta\, d\theta\, d\phi.$$

If m be not $= n$, this cannot be satisfied, unless

$$\int_0^\pi\int_0^{2\pi} Y_m\, {}_1Y_n \sin\theta\, d\theta\, d\phi = 0 \ldots\ldots\ldots\ldots (16).$$

This is the fundamental property of Laplace's coefficients.

There are some other applications of the general theorem which has been established, especially to the Theory of Electricity, which must however be left for a future opportunity.

II.—ON THE LINEAR MOTION OF HEAT.

PART II.

Let us now endeavour to find the general form of ${}_0v$, for positive and negative values of x, which is producible by any distribution of heat, an infinite time previously, or which is the same, to find the form of the function f, which renders v, or $\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} f(x + 2\alpha t^{\frac{1}{2}})$, possible for all values of x, and for all values of t, back to $-\infty$.

If v be possible for all values of t, it may be represented by

$$\Sigma\left(P_i \cos\frac{2i\pi t}{p} + Q_i \sin\frac{2i\pi t}{p}\right) \ldots\ldots\ldots (d),$$

where P_i and Q_i are functions of x, which it is our object to determine.

Modifying (a) and (b), so that the multiplier of $d\alpha\,\varepsilon^{-\alpha^2}$, in the first members, may be of the form $f(x + 2\alpha t^{\frac{1}{2}})$, and the second members of the forms $P\cos(2mt)$, $Q\sin(2mt)$, and putting $m = \frac{i\pi}{p}$, we have

$$\int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} \varepsilon^{-(x+2\alpha t^{\frac{1}{2}})\sqrt{\frac{i\pi}{p}}} \begin{matrix}\sin\\ \cos\end{matrix} \left\{\sqrt{\frac{i\pi}{p}}\,(x+2\alpha t^{\frac{1}{2}})\right\}$$
$$= \pi^{\frac{1}{2}} \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \begin{matrix}\sin\\ \cos\end{matrix} \left(x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi t}{p}\right),$$

$$\int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} \varepsilon^{(x+2\alpha t^{\frac{1}{2}})\sqrt{\frac{i\pi}{p}}} \begin{matrix}\sin\\ \cos\end{matrix} \left\{\sqrt{\frac{i\pi}{p}}\,(x+2\alpha t^{\frac{1}{2}})\right\}$$
$$= \pi^{\frac{1}{2}} \varepsilon^{x\sqrt{\frac{i\pi}{p}}} \begin{matrix}\sin\\ \cos\end{matrix} \left(x\sqrt{\frac{i\pi}{p}} + \frac{2i\pi t}{p}\right).$$

Hence we see that the most general expression for v, when it is of the form (d), is

$$v = \sum_0^\infty \left[\varepsilon^{x\sqrt{\frac{i\pi}{p}}}\left\{A_i \cos\left(x\sqrt{\frac{i\pi}{p}} + \frac{2i\pi t}{p}\right) + B_i \sin\left(x\sqrt{\frac{i\pi}{p}} + \frac{2i\pi t}{p}\right)\right\}\right.$$
$$\left. + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\left\{A_i' \cos\left(x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi t}{p}\right) + B_i' \sin\left(x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi t}{p}\right)\right\}\right] \ldots (12).$$

This then represents the most general state of the temperature of the body when the heat has been moving freely for an infinite time, and therefore, whatever be the initial distribution, the ultimate distribution must be of this form.

Putting $t = 0$, we find

$${}_0v = \sum_0^\infty \left\{\varepsilon^{x\sqrt{\frac{i\pi}{p}}}\left(A_i \cos x\sqrt{\frac{i\pi}{p}} + B_i \sin x\sqrt{\frac{i\pi}{p}}\right)\right.$$
$$\left. + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\left(A_i' \cos x\sqrt{\frac{i\pi}{p}} - B_i' \sin x\sqrt{\frac{i\pi}{p}}\right)\right\} \ldots (13),$$

for the simplest form of the distribution at any period, which is producible after an indefinite time.

This expression consists of two independent parts, one containing $\varepsilon^{x\sqrt{\frac{i\pi}{p}}}$ as a factor in each term, and the other $\varepsilon^{-x\sqrt{\frac{i\pi}{p}}}$. By examining (12), we see that the former of these gives rise to a series of *waves* of heat, proceeding in the negative direction; and the latter to a series of waves proceeding in the positive direction; and that while a wave in the former system moves from $x = \infty$ to $x = -\infty$, and a wave in the latter from $x = -\infty$ to $x = \infty$, its *amplitude* diminishes from ∞ to 0. As the two systems of waves are precisely similar, we may confine our attention to one of them, the latter for instance, which consists of waves proceeding in the positive direction

The initial distribution which gives rise to them is

$$_0v = \sum_0^\infty \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \left(A_i \cos x \sqrt{\frac{i\pi}{p}} - B_i \sin x \sqrt{\frac{i\pi}{p}} \right) \ldots (e).$$

Now it has been already shown that the initial distribution $F(-x)$ on the negative side produces the same value of v_0. On comparing the expression for $F(-x)$, given by (8), with (*e*), we see that the positive part $_0v$ in the latter is *turned over*, and added to the negative part, to make $F(-x)$. This should obviously make the values of v_0 be the same in the two cases; but the variable temperatures of every point not situated in the zero plane should be different. Hence we see how it is that the distribution, $F(-x)$, on the negative side, makes the temperature of the zero plane periodical, and therefore real for every value of t, and that of every other parallel plane, unperiodical, and impossible for negative values of t.

If, in (12), $f_1 t$ and $f_2 t$ be the parts of v_0 arising from the series, of which the coefficients are A_i, B_i, and A_i', B_i', the value of v becomes

$$\left.\begin{aligned} v = \frac{1}{p} \sum_{-\infty}^{\infty} \Big[& \varepsilon^{x\sqrt{\frac{i\pi}{p}}} \int_{-\frac{1}{2}p}^{\frac{1}{2}p} dt' f_1 t' \cos\left\{ x\sqrt{\frac{i\pi}{p}} + \frac{2i\pi t}{p}(t-t') \right\} \\ & + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \int_{-\frac{1}{2}p}^{\frac{1}{2}p} dt' f_2 t' \cos\left\{ x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi t}{p}(t-t') \right\} \Big], \\ \text{or, when } p = \infty, \qquad & \\ \pi v = \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} dt' \, [& \varepsilon^{\beta^{\frac{1}{2}}x} f_1 t' \cos\{\beta^{\frac{1}{2}}x + 2\beta(t-t')\} \\ & + \varepsilon^{-\beta^{\frac{1}{2}}x} f_2 t' \cos\{\beta^{\frac{1}{2}}x - 2\beta(t-t')\}]. \end{aligned}\right\} \ldots (14).$$

From the latter of these forms, that given by Fourier (*Théorie de la Chaleur*, p. 544,) may be readily deduced, and by putting $f_1 t' = 0$ in the former, we have the solution (given by Kelland, in his *Treatise on Heat*, p. 127,) which Fourier employed to express the diurnal and annual variations in the temperature of the earth at small depths. It is obviously suited to the case in which the temperature of the body, below the surface, is naturally constant, and all the periodical variations are produced by external causes, and proceed downwards, from the surface.

2. Let the body be supposed to be terminated by the zero plane, and to radiate heat across it, according to Newton's law; and let the external temperature be a given function, ξt, of the time. To find the state of the temperature of the

body, after any time has elapsed, the initial distribution in the body, or on the positive side of the zero plane, being ϕx.

Let the medium into which the surface radiates be supposed to be removed, and, instead of it, let the body extend infinitely on the negative side. The first thing to be done is to find the distribution on the negative side which will exactly supply the place of the radiation. The conditions which this must be chosen to satisfy, are

$$\left.\begin{aligned} \left(\frac{dv}{dx}\right)_0 &= h(v_0 - \xi t), \\ {}_0v &= \phi x, \text{ when } x \text{ is positive,} \end{aligned}\right\} \ldots \text{(a)},$$

where h is the radiating power of the surface. If, in addition to the latter of these equations, we assume ψx to be the required initial distribution on the negative side, the variable temperature, v, of any point, will be given by (4).

Differentiating this equation, and putting $x = 0$ in the result, we have

$$\pi^{\frac{1}{2}}\left(\frac{dv}{dx}\right)_0 = \int_0^\infty da\, \varepsilon^{-a^2}\phi'(2at^{\frac{1}{2}}) + \int_{-\infty}^0 da\, \varepsilon^{-a^2}\psi'(2at^{\frac{1}{2}}) + \frac{\phi o - \psi o}{2t^{\frac{1}{2}}}.$$

Now ψo must be $= \phi o$, as otherwise, at the commencement of the variation, the radiation would be infinite. Hence we have, from (a),

$$\int_0^\infty da\, \varepsilon^{-a^2}\phi'(2at^{\frac{1}{2}}) + \int_{-\infty}^0 da\, \varepsilon^{-a^2}\psi'(2at^{\frac{1}{2}})$$
$$= h\left\{\int_0^\infty da\, \varepsilon^{-a^2}\phi(2at^{\frac{1}{2}}) + \int_{-\infty}^0 da\, \varepsilon^{-a^2}\psi(2at^{\frac{1}{2}}) - \pi^{\frac{1}{2}}\xi t\right\},$$

or $\displaystyle\int_0^\infty da\, \varepsilon^{-a^2}[\phi'(2at^{\frac{1}{2}}) + \psi'(-2at^{\frac{1}{2}}) - h\{\phi(2at^{\frac{1}{2}}) + \psi(-2at^{\frac{1}{2}})\}] = -\pi^{\frac{1}{2}}h\xi t.$

Hence, if

$$\xi t = \Sigma\left(A_i \cos\frac{2i\pi t}{p} + B_i \sin\frac{2i\pi t}{p}\right) \ldots \text{(b)},$$

and if Fx be determined by (8), then, using x instead of $2at^{\frac{1}{2}}$, we must have

$$\phi' x + \psi'(-x) - h\{\phi x + \psi(-x)\} = -hFx \;.\;.\; \text{(c)},$$

$$\text{or} \quad \frac{d\psi(-x)}{-dx} - h\psi(-x) = h(\phi x + Fx) - \frac{d\phi x}{dx};$$

$$\left.\begin{aligned} \text{therefore } \psi(-x) &= -\varepsilon^{-hx}\int \varepsilon^{hx}\left\{h(\phi x - Fx) - \frac{d\phi x}{dx}\right\}dx \\ \text{or } \psi(-x) &= \phi x - h\,\varepsilon^{-hx}\int \varepsilon^{hx}(2\phi x - Fx)\,dx, \end{aligned}\right\} \ldots (15).$$

The function ψ being determined from this, the solution of the problem is found by using the result in (4).

After the motion has continued for a long period of time, the irregularities of the initial distribution disappear, and the variations of the temperature of the body are reduced, by the periodical variations of the external temperature, to a permanently periodical state. Let us suppose that this permanent state has been reached when $t=0$. That this may be the case, we must choose ϕx of such a form, that ψx when x is negative, and ϕx when x is positive, may make ${}_0v$ be of the form (e), No. 1. Let us therefore assume

$$\left.\begin{array}{l}\text{when } x \text{ is negative } \psi x = \\ \text{when } x \text{ is positive } \phi x = \end{array}\right\} \Sigma \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\left(a_i \cos x\sqrt{\frac{i\pi}{p}} - b_i \sin x\sqrt{\frac{i\pi}{p}}\right),$$

where a_i and b_i are to be determined so as to satisfy (c).

Using these values of ψx and ϕx in (c), and using for Fx its value (8), we have, by equating the coefficients of

$$\left(\varepsilon^{x\sqrt{\frac{i\pi}{p}}} + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\right)\cos x\sqrt{\frac{i\pi}{p}}, \text{ and } \left(\varepsilon^{x\sqrt{\frac{i\pi}{p}}} - \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\right)\sin x\sqrt{\frac{i\pi}{p}},$$

in the two members of the resulting equation,

$$a_i\left(h + \sqrt{\frac{i\pi}{p}}\right) + b_i\sqrt{\frac{i\pi}{p}} = hA_i,$$

$$a_i\sqrt{\frac{i\pi}{p}} - b_i\left(\sqrt{\frac{i\pi}{p}} + h\right) = -hB_i;$$

$$\text{whence}\quad \left.\begin{array}{l} a_i = \dfrac{h\left\{A_i\left(\sqrt{\frac{i\pi}{p}} + h\right) - B_i\sqrt{\frac{i\pi}{p}}\right\}}{h^2 + 2h\sqrt{\frac{i\pi}{p}} + 2\frac{i\pi}{p}} \\[2ex] b_i = \dfrac{h\left\{A_i\sqrt{\frac{i\pi}{p}} + B_i\left(\sqrt{\frac{i\pi}{p}} + h\right)\right\}}{h^2 + 2h\sqrt{\frac{i\pi}{p}} + 2\frac{i\pi}{p}} \end{array}\right\} \ldots . (16).$$

If, in (12), we put A_i and B_i each equal to zero, and use these values of a_i and b_i instead of A_i' and B_i', the resulting expression is the variable temperature of any point. Let, for brevity,

$$D_i = \sqrt{\left(h^2 + 2h\sqrt{\frac{i\pi}{p}} + 2\frac{i\pi}{p}\right)},$$

$$\cos\delta_i = \frac{\sqrt{\frac{i\pi}{p}} + h}{D_i}, \qquad \sin\delta_i = \frac{\sqrt{\frac{i\pi}{p}}}{D_i}.$$

Hence, using for A_i and B_i, their values, which satisfy (b), we have

$$pv = h\sum_{-\infty}^{\infty} \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \int_{-\frac{1}{2}p}^{\frac{1}{2}p} dt' \, \Xi t' \, \frac{\cos\left\{x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi}{p}(t-t') + \delta_i\right\}}{D_i} \quad . . (17),$$

which agrees with the expression given by Poisson, in p. 431 of his *Théorie de la Chaleur*.

N. N.

III.—ON THE INTERSECTION OF CURVES.

By Arthur Cayley, B.A. Fellow of Trinity College.

The following theorem is quoted in a note of Chasles' Aperçu Historique, &c. *Memoires de Bruxelles,* tom. XI. p. 149, where M. Chasles employs it in the demonstration of Pascal's theorem. "If a curve of the third order pass through eight of the points of intersection of two curves of the third order, it passes through the ninth point of intersection." The application in question is so elegant, that it deserves to be generally known. Consider a hexagon inscribed in a conic section. The aggregate of three alternate sides may be looked upon as forming a curve of the third order, and that of a remaining sides, a second curve of the same order. These two intersect in nine points, viz. the six angular points of the hexagon, and the three points which are the intersections of pairs of opposite sides. Suppose a curve of the third order passing through eight of these points, viz. the aggregate of the conic section passing through the angular points of the hexagon, and of the line forming two of the three intersections of pairs of opposite sides. This passes through the ninth point, by the theorem of Chasles, *i. e.* the three intersections of pairs of opposite sides lie in the same straight line, (since obviously the third intersection does *not* lie in the conic section), which is Pascal's theorem.

The demonstration of the above property of curves of the third order is one of extreme simplicity. Let $U=0$, $V=0$, be the equations of two curves of the third order, the curve of the same order which passes through eight of their points of intersection, (which may be considered as eight perfectly arbitrary points), and a ninth arbitrary point, will be perfectly determinate. Let U_0, V_0, be the values of U, V, when the co-ordinates of this last point are written in place of x, y. Then $UV_0 - U_0V = 0$, satisfies the above conditions, or it is the equation to the curve required; but it is an equation

which is satisfied by all the nine points of intersection of the two curves, *i.e.* any curve that passes through eight of these points of intersection, passes also through the ninth.

Consider generally two curves, $U_m = 0$, $V_n = 0$, of the orders m and n respectively, and a curve of the r^{th} order (r not less than m or n) passing through the mn points of intersection. The equation to such a curve will be of the form

$$U = u_{r-m} U_m + v_{r-n} V_n = 0,$$

u_{r-m}, v_{r-n}, denoting two polynomes of the orders $r-m$, $r-n$, with all their coefficients complete. It would at first sight appear that the curve $U = 0$ might be made to pass through as many as $\{1 + 2 \ldots + (r - m + 1)\} + \{1 + 2 \ldots (r - n + 1)\} - 1$, arbitrary points, *i.e.*

$$\tfrac{1}{2}(r-m+1)(r-m+2) + \tfrac{1}{2}(r-n+1)(r-n+2) - 1;$$

or, what is the same thing,

$$\tfrac{1}{2}r.(r+3) - mn + \tfrac{1}{2}(r-m-n+1)(r-m-n+2)$$

arbitrary points, such being apparently the number of disposable constants. This is in fact the case as long as r is not greater than $m+n-1$; but when r exceeds this, there arise, between the polynomes which multiply the disposable coefficients, certain linear relations, which cause them to group themselves into a smaller number of disposable quantities. Thus, if r be not less than $m+n$, forming different polynomes of the form $x^\alpha y^\beta . V_n$ [$\alpha + \beta =$ or $< m$], and multiplying by the coefficients of $x^\alpha y^\beta$ in U_m, and adding, we obtain a sum $U_m V_n$, which might have been obtained by taking the different polynomes of the form $x^\gamma y^\delta . U_m$ [$\gamma + \delta =$ or $< n$], multiplying by the coefficients of $x^\gamma y^\delta$ in V_n, and adding. Or we have a linear relation between the different polynomes of the forms $x^\alpha y^\beta V_n$, and $x^\gamma y^\delta U_m$. In the case where r is not less than $m+n+1$, there are two more such relations, viz. those obtained in the same way from the different polynomes $x^\alpha y^\beta . xV_n$, $x^\gamma y^\delta . xU_m$, and $x^\alpha y^\beta . yV_n$, $x^\gamma y^\delta . yU_m$, &c.; and in general, whatever be the excess of r above $m+n+1$, the number of these linear relations is

$$1 + 2 \ldots (r-m-n+1) = \tfrac{1}{2}(r-m-n+1)(r-m-n+2).$$

Hence, if r be not less than $m+n$, the number of points through which a curve of the r^{th} order may be made to pass, in addition to the mn points which are the intersections of $U_m = 0$, $V_n = 0$, is simply $\tfrac{1}{2}r(r+3) - mn$. In the case of $r = m+n-1$, or $r = m+n-2$, the two formulæ coincide. Hence we may enunciate the theorem—

"A curve of the r^{th} order, passing through the mn points of intersection of two curves of the m^{th} and n^{th} orders re-

spectively, may be made to pass through $\frac{1}{2}r(r+3)-mn+\frac{1}{2}(m+n-r-1)(m+n-r-2)$ arbitrary points, if r be not greater than $m+n-3$: if r be greater than this value, it may be made to pass through $\frac{1}{2}r(r+3)-mn$ points only."

Suppose r not greater than $m+n-3$, and a curve of the r^{th} order made to pass through

$$\tfrac{1}{2}r(r+3)-mn+\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

arbitrary points, and

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of the mn points of intersection above. Such a curve passes through $\frac{1}{2}r(r+3)$ given points, and though the $mn-\frac{1}{2}(m+n-r-1)(m+n-r-2)$ latter points are not perfectly arbitrary, there appears to be no reason why the relation between the positions of these points should be such, as to prevent the curve from being *completely determined* by these conditions. But if it be so, it must pass through the remaining $\frac{1}{2}(m+n-r-1)(m+n-r-2)$ points of intersection, or we have the theorem—

"If a curve of the r^{th} order (r not less than m or n, not greater than $m+n-3$) pass through

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of the points of intersection of two curves of the m^{th} and n^{th} orders respectively, it passes through the remaining

$$\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

points of intersection."

IV.—RESEARCHES IN ROTATORY MOTION.

By Andrew Bell.

1. This article contains some theorems in rotatory motion, respecting the effect of the centrifugal force arising from the rotation of a body about an axis, in producing rotation about another axis inclined at any angle to the former.

To avoid unnecessary circumlocution, I propose that, instead of the expression—the effect of the centrifugal force arising from a rotation about an axis in producing rotation about another axis, this more concise one should be used, namely, the centrifugal effect for one axis about another axis.

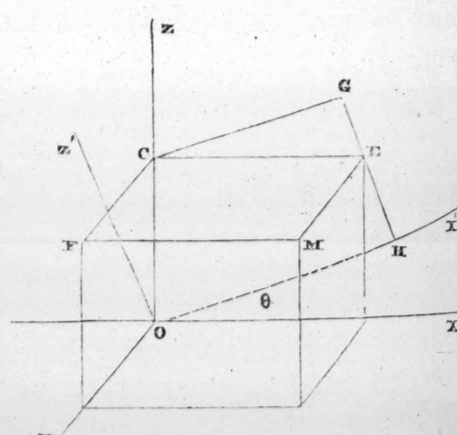
2. The centrifugal effects of any rotations for two rectangular axes about the third co-ordinate rectangular axis, is equal to the similar effect for either of these axes, when the rotation about it is equal to the resultant of the rotations about them both.

Let (x'), (y'), (z'), denote the three rectangular axes; q, r, the respective rotations about the two latter; ω' their resultant rotation about an axis (z) in their plane; and θ the inclination of (z) and (z').

The sum of the centrifugal effects for the axes (y'), (z'), about (x'), is
$$= (q^2 + r^2)\ \Sigma y'z'\Delta m,$$
Δm denoting an element of the body.

But $q = \omega' \sin\theta$, and $r = \omega' \cos\theta$; and hence this effect becomes
$$= \omega'^2 \Sigma y'z'\Delta m;$$
which is the similar effect of an angular velocity ω' about either of the axes (y'), (z').

3. If to the second of two given axes* having any inclination, a third axis is drawn perpendicularly and in the same plane with the given axes; and if any given rotation about the first of the two given axes is decomposed into its constituents about the other two axes, then the centrifugal effect for the first axis about the second, is equal to the similar effect of the constituent rotation about the third axis.



Let Ox, Oy, Oz, be three rectangular axes, and Ox', Oz', perpendicular axes in the plane of (x) and (z); then if a rotation about (z) is decomposed into its constituents about (x') and (z'), the centrifugal effect for (z) about (x') is equal to that for (z') about (x').

If ω is the angular velocity about (z), then the centrifugal effect arising from this motion produces a force which may be represented in direction and quantity by CM, acting on an element Δm at M, and which can be resolved into CE and CF, so that the component forces parallel to the axes (x) and (y), are respectively
$$\omega^2 x\Delta m \text{ and } \omega^2 y\Delta m.$$

But CE may be resolved into CG parallel to (x'), and GE parallel to (z'), of which the former has no effect round (x');

* The axes concerned in any of these theorems are to be understood as co-originate, that is, as having the same origin.

and $GE = CE \sin\theta$, if θ is the inclination of (x) to (x'). Hence the forces acting on Δm that produce rotation about (x'), are,

$$CF \text{ or } EM \text{ parallel to } (y), \text{ and } = \omega^2 y \Delta m,$$

$$\text{and } GE \text{ parallel to } (z'), \text{ and } = \omega^2 \sin\theta x \Delta m.$$

The former of these forces acts at the extremity of an ordinate $z' = EH$, and the latter at that of an ordinate $y = EM$. Hence, as they conspire, the sum of their momenta about (x') is

$$= \omega^2 \Sigma y z' \Delta m + \omega^2 \sin\theta \, \Sigma x y \Delta m.$$

But $x = x' \cos\theta - z' \sin\theta$, and hence

$$\omega^2 \sin\theta \, \Sigma x y \Delta m = \omega^2 \sin\theta \cos\theta \, \Sigma x' y \Delta m - \omega^2 \sin^2\theta \, \Sigma y z' \Delta m.$$

And since $\Sigma x' y \Delta m$ can have no effect about (x'), therefore the effect round (x') is

$$= \omega^2 (1 - \sin^2\theta) \, \Sigma y z' \Delta m = \omega^2 \cos^2\theta \, \Sigma y z' \Delta m.$$

Now if ω' is the constituent rotation about (z'), resulting from the decomposition of the rotation about (z) into its constituents about (z') and (x'); then the centrifugal effect for (z') about (x') is $= \omega'^2 \Sigma y z' \Delta m$.

And since $\omega' = \omega \cos\theta$, the latter expression becomes

$$= \omega^2 \cos^2\theta \, \Sigma y z' \Delta m,$$

which is the same as the above expression.

4. The effect of the centrifugal force for any axis about another, whatever be their inclination, is the same as its similar effect for an axis perpendicular to the first, and in the same plane with these two axes, reduced to its constituent effect about the second axis; considering these effects as measured by momenta, applied at a given distance from these axes.

Let Oz and Ox' be any two axes, and Ox another axis perpendicular to the first, and the rotation about Oz be denoted by ω.

The centrifugal effect for (z) about (x') was already proved to be (3) $= \omega^2 \cos^2\theta \, \Sigma y z' \Delta m$,

ω being the angular velocity about (z). But the similar effect for (z) about (x) is

$$= \omega^2 \Sigma y z \Delta m;$$

and this effect being decomposed into its constituents about (x') and (z'), gives for the former

$$\omega^2 \cos\theta \, \Sigma y z \Delta m.$$

And since $z = z' \cos\theta + x' \sin\theta$, therefore this constituent

$$= \omega^2 \cos^2\theta\ \Sigma yz'\Delta m + \omega^2 \sin\theta\cos\theta\ \Sigma yx'\Delta m;$$

and since $\Sigma yx'\Delta m$ can produce no effect round (x'), the latter expression becomes

$$\omega^2 \cos^2\theta\ \Delta yz'\Delta m,$$

which is the same as the above result.

V.—ON THE TRANSFORMATION OF DEFINITE INTEGRALS.

By George Boole.

1. Jacobi, in the 15th vol. of *Crelle's Journal*, has proved the following remarkable transformation of a definite integral, quoted, with a demonstration, in Mr. Gregory's *Examples of the Diff. and Integ. Calculus*, p. 497, viz.

$$\int_0^\pi dx\, f^{(r)}(\cos x)(\sin x)^{2r} = 1.3.5..(2r-1)\int_0^\pi dx\, f(\cos x)\cos rx,$$

where $f^{(r)}(z) = \left(\dfrac{d}{dz}\right)^r f(z)$, and all the differential coefficients up to the $(r-1)^{\text{th}}$ inclusive, remain continuous from $x=0$ to $x=\pi$. In the following pages I purpose to offer some development of a principle in analysis, which, while capable of being applied to various other questions, leads also to the theory of a class of transformations, of which the above is but a particular example.

2. By the well-known relation connecting the first and second of the Eulerian integrals,

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} = \frac{\Gamma(l')\,\Gamma(m')}{\Gamma(l'+m')};$$

hence

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(l'+m')}{\Gamma(l')\,\Gamma(m')\,\Gamma(l+m)}\int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} \ldots (2),$$

l, m, l', m', being positive constant quantities. The principle on which our investigation will rest is simply this,—that the theorem (2) remains equally true, whether l, m, l', m', represent constant quantities, or symbols of operation, combining according to the same laws, and admitting, under the particular conditions of the question, of the same interpretation.

3. Let us now consider the definite integral,

$$\int_0^1 dz\, z^{l-1}\,(1-z)^{m-1}\,f(z).$$

This may be written under the form $\int_0^1 dz\, z^{l-1}\,(1-z)^{m-1}\,f(\varepsilon^{\theta} z)$, provided that $\varepsilon^{\theta}=1$. Let $z=\varepsilon^{\phi}$, then

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(\varepsilon^{\theta} z) = \int_0^1 dz\, z^{l-1}\,(1-z)^{m-1} f(\varepsilon^{\theta+\phi})$$

$$= \int_0^1 dz\, z^{l-1}\,(1-z)^{m-1}\,\varepsilon^{\phi\frac{d}{d\theta}} f(\varepsilon^{\theta}),$$

by Taylor's theorem. Replacing ε^{ϕ} by z, this gives

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(\varepsilon^{\theta} z) = \int_0^1 dz\, z^{\frac{d}{d\theta}+l-1}\,(1-z)^{m-1} f(\varepsilon^{\theta})\ldots.(3).$$

Similarly we have

$$\int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} f(\varepsilon^{\theta} z) = \int_0^1 dz\, z^{\frac{d}{d\theta}+l'-1}\,(1-z)^{m'-1} f(\varepsilon^{\theta}).$$

Now by theorem (2), and under the limitations implied in the principle above enunciated, the right-hand members of these two equations are connected by the relation

$$\int_0^1 dz\, z^{\frac{d}{d\theta}+l-1}\,(1-z)^{m-1} f(\varepsilon^{\theta})$$

$$= \frac{\Gamma\left(\frac{d}{d\theta}+l\right)\Gamma(m)\,\Gamma\left(\frac{d}{d\theta}+l'+m'\right)}{\Gamma\left(\frac{d}{d\theta}+l'\right)\Gamma(m')\,\Gamma\left(\frac{d}{d\theta}+l+m\right)} \int_0^1 dz\, z^{\frac{d}{d\theta}+l'-1}\,(1-z)^{m'-1} f(\varepsilon^{\theta})\ldots(4);$$

hence, putting the first members respectively in the room of the second,

$$\int_0^1 dz\, z^{l-1}\,(1-z)^{m-1} f(\varepsilon^{\theta} z)$$

$$= \frac{\Gamma\left(\frac{d}{d\theta}+l\right)\Gamma(m)\,\Gamma\left(\frac{d}{d\theta}+l'+m'\right)}{\Gamma\left(\frac{d}{d\theta}+l'\right)\Gamma(m')\,\Gamma\left(\frac{d}{d\theta}+l+m\right)} \int_0^1 dz\, z^{l'-1}\,(1-z)^{m'-1} f(\varepsilon^{\theta} z)$$

$$= \int_0^1 dz\, z^{l'-1}(1-z)^{m'-1}\, \frac{\Gamma\left(\frac{d}{d\theta}+l\right)\Gamma(m)\,\Gamma\left(\frac{d}{d\theta}+l'+m'\right)}{\Gamma\left(\frac{d}{d\theta}+l'\right)\Gamma(m')\,\Gamma\left(\frac{d}{d\theta}+l+m\right)} f(\varepsilon^{\theta} z)\ldots(5).$$

4. Now to shew the conditions under which $\dfrac{d}{d\theta}$ admits of the required interpretation, we observe, that if $f(z)$ can be developed in positive ascending powers of z, we shall have, on effecting the development in the second member of the above equation, a series of terms of the form

$$A_i \int_0^1 dz\, z^{i+l'-1}(1-z)^{m'-1} \frac{\Gamma\left(\frac{d}{d\theta}+l\right)\Gamma(m)\,\Gamma\left(\frac{d}{d\theta}+l'+m'\right)}{\Gamma\left(\frac{d}{d\theta}+l'\right)\Gamma(m')\,\Gamma\left(\frac{d}{d\theta}+l+m\right)}\,\varepsilon^{i\theta} \ldots (6);$$

to each of which there will exist a corresponding term with the same constant coefficient, A_i, in the development of the first member. But by a known theorem,

$$\psi\left(\frac{d}{d\theta}\right)\varepsilon^{i\theta} = \psi(i)\,\varepsilon^{i\theta};$$

hence in each term the symbol $\dfrac{d}{d\theta}$ may be interpreted by a positive constant quantity, and the theorem verified by successive applications of (2). It is therefore true for all forms of $f(z)$ which can be expanded without negative indices.

In (5) again write $f(\varepsilon^{\theta+\phi})$ for $f(\varepsilon^{\theta} z)$, and observing that

$$\psi\left(\frac{d}{d\theta}\right) f(\varepsilon^{\theta+\phi}) = \psi\left(\frac{d}{d\phi}\right) f(\varepsilon^{\theta+\phi}) = \psi\left(\frac{d}{d\phi}\right) f(\varepsilon^{\phi}),$$

since $\varepsilon^{\theta} = 1$, we have

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(z)$$

$$= \int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} \frac{\Gamma\left(\frac{d}{d\phi}+l\right)\Gamma(m)\,\Gamma\left(\frac{d}{d\phi}+l'+m'\right)}{\Gamma\left(\frac{d}{d\phi}+l'\right)\Gamma(m')\,\Gamma\left(\frac{d}{d\phi}+l+m\right)} f(\varepsilon^{\phi}) \ldots (7),$$

with the relation $\varepsilon^{\phi} = z$.

5. In applying the above theorem, let us first suppose that $l - l'$ and $m - m'$ are positive integers; then from the known relations,

$$\psi\left(\frac{d}{d\phi}+a\right) f(\varepsilon^{\phi}) = \varepsilon^{-a\phi}\,\psi\left(\frac{d}{d\phi}\right)\varepsilon^{a\phi} f(\varepsilon^{\phi}) \ldots . (8),$$

$$\frac{d}{d\phi}\left(\frac{d}{d\phi}-1\right)..\left(\frac{d}{d\phi}-m+1\right) f(\varepsilon^{\phi}) = z^m \left(\frac{d}{dz}\right)^m f(z) \ldots . (9),$$

we easily find

$$\frac{\Gamma\left(\frac{d}{d\phi}+l\right)}{\Gamma\left(\frac{d}{d\phi}+l'\right)} f(\varepsilon^{\phi}) = \varepsilon^{-(l-1)\phi} \frac{\Gamma\left(\frac{d}{d\phi}+1\right)}{\Gamma\left\{\frac{d}{d\phi}-(l-l')+1\right\}} f(\varepsilon^{\phi})$$

$$= \varepsilon^{-(l-1)\phi} \frac{d}{d\phi}\left(\frac{d}{d\phi}-1\right)\cdot\cdot\left\{\frac{d}{d\phi}-(l-l')+1\right\} f(\varepsilon^{\phi})$$

$$= z^{-(l-1)} z^{l-l'} \left(\frac{d}{dz}\right)^{l-l'} z^{l-1} f(z)$$

$$= z^{-(l'-1)} \left(\frac{d}{dz}\right)^{l-l'} z^{l-1} f(z) \;\ldots\ldots\ldots\; (10);$$

and, by similar reasoning,

$$\frac{\Gamma\left(\frac{d}{d\phi}+l'+m'\right)}{\Gamma\left(\frac{d}{d\phi}+l+m\right)} f(\varepsilon^{\phi}) = z^{-(l+m-1)} \left(\frac{d}{dz}\right)^{l'+m'-l-m} z^{l'+m'-1} f(z) \ldots (11).$$

Substitute these forms in (7), and there will result

$$\int_0^1 dz\, z^{l-1} (1-z)^{m-1} f(z)$$

$$= \frac{\Gamma(m)}{\Gamma(m')} \int_0^1 dz\,(1-z)^{m'-1} \left(\frac{d}{dz}\right)^{l-l'} z^{-m} \left(\frac{d}{dz}\right)^{l'+m'-l-m} z^{l'+m'-1} f(z) \ldots (12),$$

a transformation of great generality, from which many particular results of an interesting character may be obtained.

It is obvious that the above theorem is equally true, when one or both the indices of $\frac{d}{dz}$ are fractional; but in this case it would be necessary, in the interpretation of our symbols, to revert to (7).

6. When the index of $\frac{d}{dz}$ is a negative integer, it must be observed that the symbol $\left(\frac{d}{dz}\right)^{-1}$ applied to a function developable in ascending positive powers of z, is equivalent to integration between the limits 0 and z; but if it should happen, which in the above theorem it cannot, that the development involves only negative powers of z, then the limits are $\pm\,\infty$, and z; for $\left(\frac{d}{dz}\right)^{-1}$ is evidently to be interpreted as the inverse of $\frac{d}{dz}$: now $\frac{d}{dz} z^m = mz^{m-1}$, therefore $\left(\frac{d}{dz}\right)^{-1} mz^{m-1} = z^m$, which is only true on the above assumption relative to the limits.

7. As particular illustrations of (12), let $l = l' = m' = \frac{1}{2}$, m being any positive constant, then

$$\int_0^1 dz\, z^{-\frac{1}{2}} (1-z)^{m-1} f(z) = \frac{\Gamma(m)}{\Gamma(\frac{1}{2})} \int_0^1 dz\, z^{-m} (1-z)^{-\frac{1}{2}} \left(\frac{d}{dz}\right)^{-(m-\frac{1}{2})} f(z).$$

As the index of $\dfrac{d}{dz}$ must be an integer, let $m - \frac{1}{2} = r$, we have

$$\int_0^1 dz\, z^{-\frac{1}{2}} (1-z)^{r-\frac{1}{2}} f(z) = \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} \int_0^1 dz\, z^{-r+\frac{1}{2}} (1-z)^{-\frac{1}{2}} \left(\frac{d}{dz}\right)^{-r} f(z).$$

Put $z = (\cos x)^2$, then

$$(1-z) = (\sin x)^2, \text{ and } -\tfrac{1}{2} dz\, z^{-\frac{1}{2}} (1-z)^{\frac{1}{2}} = dx;$$

the limits inverted are 0 and $\frac{1}{2}\pi$, whence

$$\int_0^{\frac{1}{2}\pi} dx\, (\sin x)^{2r} f\{(\cos x)^2\} = \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} \int_0^{\frac{1}{2}\pi} dx\, \frac{\left(\frac{d}{dz}\right)^{-r} f(z)}{(\cos x)^{2r}} \ldots (13),$$

provided that the integrations implied by the symbol $\left(\dfrac{d}{dz}\right)^{-r}$, be taken between the limits 0 and $(\cos x)^2$, $f(z)$ being developable in ascending positive powers of z.

Another form of the theorem will be obtained by assuming $\dfrac{1}{z^r} \left(\dfrac{d}{dz}\right)^{-r} f(z) = F(z)$, then

$$\int_0^{\frac{1}{2}\pi} dx\, (\sin x)^{2r} \left(\frac{d}{dz}\right)^r \{z\, F(z)\} = \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} \int_0^{\frac{1}{2}\pi} dx\, F\{(\cos x)^2\} \ldots (14),$$

provided that after the differentiations are performed we change z into $(\cos x)^2$, and that $F(z)$ have no negative indices in its development, and, with its differential coefficients up to the $(r-1)^{\text{th}}$, is continuous within the limits of integration.

8. Again, in (12) let $l' = m$, $m' = l$, we have

$$\int_0^1 dz\, z^{l-1} (1-z)^{m-1} f(z) = \frac{\Gamma(m)}{\Gamma(l)} \int_0^1 dz\, (1-z)^{l-1} \left(\frac{d}{dz}\right)^{l-m} z^{l-1} f(z),$$

put $z^{l-1} f(z) = F(z)$, then

$$\int_0^1 dz\, (1-z)^{m-1} F(z) = \frac{\Gamma(m)}{\Gamma(l)} \int_0^1 dz\, (1-z)^{l-1} \left(\frac{d}{dz}\right)^{l-m} F(z).$$

If we further make $m = 1$, and for $l - 1$ write l, we find, on transposing the members,

$$\int_0^1 dz\, (1-z)^l \left(\frac{d}{dz}\right)^l F(z) = \Gamma(l+1) \int_0^1 dz\, F(z).$$

From the relation between $f(z)$ and $F(z)$, it appears that the latter function cannot involve in its development terms whose indices are lower than l. If any such occur, they

must be considered separately. In this way we find for the complete form of the theorem,

$$\int_0^1 dz\,(1-z)^l \left(\frac{d}{dz}\right)^l F(z)$$

$$=\Gamma(l+1)\left\{-F(o)-\frac{F'(o)}{1.2}\;\ldots-\frac{F^{(l-1)}(o)}{1.2\ldots l}+\int_0^1 dz\,F(z)\right\}\;\ldots(15),$$

true whenever $F(z)$ is a function developable by Taylor's theorem.

9. When $l-l'$, and $m-m'$ are not integers, the theorem (7) must be transformed by the introduction of additional symbols of definite integration. The general result is inconveniently long, but the method to be pursued will be sufficiently illustrated by the following examples.

Let $l=l'=m'=\frac{1}{2}$, and m be any positive constant, we have

$$\int_0^1 dz\,z^{-\frac{1}{2}}(1-z)^{m-1}f(\epsilon^\phi)=\int_0^1 dz\,z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}\frac{\Gamma\left(\frac{d}{d\phi}+1\right)}{\Gamma\left(\frac{d}{d\phi}+m+\frac{1}{2}\right)}f(\epsilon^\phi)\ldots(16),$$

$$\int_0^1 dz\,z^{-\frac{1}{2}}(1-z)^{m-1}\,\Gamma\left(\frac{d}{d\phi}+m+\frac{1}{2}\right)f(\epsilon^\phi)$$

$$=\int_0^1 dz\,z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}\,\Gamma\left(\frac{d}{d\phi}+1\right)f(\epsilon^\phi)\;\ldots(17).$$

Now $\Gamma\left(\frac{d}{d\phi}+m+\frac{1}{2}\right)f(\epsilon^\phi)=\int_0^\infty dv\,\epsilon^{-v}\,v^{\left(\frac{d}{d\phi}+m-\frac{1}{2}\right)}f(\epsilon^\phi)$, but $v^{\frac{d}{d\phi}}f(\epsilon^\phi)=f(\epsilon^\phi v)=f(vz)$; therefore

$$\Gamma\left(\frac{d}{d\phi}+m+\frac{1}{2}\right)f(\epsilon^\phi)=\int_0^\infty dv\,\epsilon^{-v}\,v^{m-\frac{1}{2}}f(vz).$$

Similarly $\quad\Gamma\left(\frac{d}{d\phi}+1\right)f(\epsilon^\phi)=\int_0^\infty dv\,\epsilon^{-v}f(vz);$

whence, substituting in (16),

$$\int_0^\infty dv\int_0^1 dz\,\epsilon^{-v}\,v^{m-\frac{1}{2}}\,z^{-\frac{1}{2}}(1-z)^{m-1}f(vz)$$

$$=\frac{\Gamma(m)}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty dv\int_0^1 dz\,\epsilon^{-v}\,z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}f(vz).$$

Put, as before, $m-\frac{1}{2}=r$ and $z=(\cos x)^2$, then

$$\int_0^\infty dv\int_0^{\frac{1}{2}\pi} dx\,\epsilon^{-v}\,v^r(\sin x)^{2r}f(v\cos x^2)$$

$$=\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty dv\int_0^{\frac{1}{2}\pi} dx\,\epsilon^{-v}f\{v(\cos x)^2\}\;\ldots.\;(18),$$

which is true for all values of r from $-\frac{1}{2}$ to ∞.

Again, in (7) let $l' = m$, $m' = l = 1$, and proceeding as above, we finally obtain

$$\int_0^\infty dv \int_0^1 dz\, \varepsilon^{-v} (v - vz)^{m-1} f(vz) = \Gamma(m) \int_0^\infty dv \int_0^1 dz\, \varepsilon^{-v} z^{m-1} f(vz) \ldots (19).$$

Both the preceding cases admit of another and more convenient form of solution, which I shall exemplify in the first. Thus, in (16),

$$\frac{\Gamma\left(\dfrac{d}{d\phi} + 1\right)}{\Gamma\left(\dfrac{d}{d\phi} + m + \frac{1}{2}\right)} f(\varepsilon^{\phi}) = \frac{1}{\Gamma(m - \frac{1}{2})} \frac{\Gamma\left(\dfrac{d}{d\phi} + 1\right)\Gamma(m - \frac{1}{2})}{\Gamma\left(\dfrac{d}{d\phi} + m + \frac{1}{2}\right)} f(\varepsilon^{\phi}) \ldots (20)$$

$$= \frac{1}{\Gamma(m - \frac{1}{2})} \int_0^1 dv\, v^{\frac{d}{d\phi}} (1 - v)^{m - \frac{3}{2}} f(\varepsilon^{\phi}) \text{ by } (1),$$

$$= \frac{1}{\Gamma(m - \frac{1}{2})} \int_0^1 dv\, (1 - v)^{m - \frac{3}{2}} f(vz);$$

whence, on substitution,

$$\int_0^1 dz\, z^{-\frac{1}{2}} (1 - z)^{m-1} f(z)$$

$$= \frac{\Gamma(m)}{\Gamma(\frac{1}{2})\,\Gamma(m - \frac{1}{2})} \int_0^1 dv \int_0^1 dz\, (1 - v)^{m - \frac{3}{2}} z^{-\frac{1}{2}} (1 - z)^{-\frac{1}{2}} f(vz):$$

put $m - \frac{1}{2} = r$, $v = (\cos y)^2$, $z = (\cos x)^2$, we find

$$\int_0^{\frac{1}{2}\pi} dx\, (\sin x)^{2r} f\{(\cos x)^2\}$$

$$= -\frac{2\Gamma(r + \frac{1}{2})}{\Gamma(r)\,\Gamma(\frac{1}{2})} \int_0^{\frac{1}{2}\pi} dx \int_0^{\frac{1}{2}\pi} dy\, \cos y\, (\sin y)^{2r-1} f\{(\cos x \cos y)^2\}.$$

This may evidently be written under the form

$$\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} dx\, dy\, \cot y\, (\sin y)^{2r} f(\cos x \cos y)$$

$$= -\frac{\Gamma(r)\,\Gamma(\frac{3}{2})}{\Gamma(r + \frac{1}{2})} \int_0^{\frac{1}{2}\pi} dx\, (\sin x)^{2r} f(\cos x)$$

$$= -\frac{\Gamma(r)\,\Gamma(\frac{3}{2})}{\Gamma(r + \frac{1}{2})} \int_0^{\frac{1}{2}\pi} dy\, (\sin y)^{2r} f(\cos y). \ldots (21).$$

In a similar way we shall find

$$\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} dx\, dy\, (\cos x)^{2r} \tan x\, f(\sin x \sin y)$$

$$= \frac{\Gamma(r)\,\Gamma(\frac{3}{2})}{\Gamma(r + \frac{1}{2})} \int_0^{\frac{1}{2}\pi} dx\, (\cos x)^{2r} f(\sin x). \ldots (22).$$

In the last theorem let $r=\frac{1}{2}$, and in the second member let $\sin x = v$, then the transformed limits being 0 and 1,

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} dx\, dy \sin x\, f(\sin x \sin y) = \tfrac{1}{2}\pi \int_0^1 dv\, f(v) \ldots . (23),$$

a very remarkable theorem, enabling us at once to assign the values of a large number of definite double integrals. It must be observed that, as before, $f(v)$ must not involve in its development negative powers of v.

10. Many particular results of great interest may be obtained without the aid of the general theorem (7). Thus from the first Eulerian integral, $\int_0^\infty \varepsilon^{-x} x^{n-1}\, dx = \Gamma(n)$, treated by the method of section 3, we find

$$\int_0^\infty dx\, \varepsilon^{-x}\left(\frac{d}{dx}\right)^{-r} F(x) = \int_0^\infty dx\, \varepsilon^{-x} F(x),$$

a theorem true for all positive values of r. If r be an integer, and $F(x)$ a function developable by Taylor's theorem, we find, by reasoning similar to that of section 8,

$$\int_0^\infty dx\, \varepsilon^{-x} F^{(r)}x = -\{F(o)+F^{(1)}(o)\ldots+F^{(r-1)}(o)\}+\int_0^\infty dx\, \varepsilon^{-x} F(x) \ldots (24).$$

From the definite integral $\int_0^\pi dx\, \varepsilon^{ax}(\cos x)^n$, we obtain the remarkable theorems

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} dx\, (\cos x)^{2r}\, F(x) = \Gamma(2r+1)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} dx\, \phi(x) \ldots . (25),$$

$$\int_0^\pi dx\, (\sin x)^{2r}\, F(x) = \Gamma(2r+1)\int_0^\pi dx\, \phi(x) \ldots . (26),$$

where $F(x) = \left(\frac{d^2}{dx^2}+2^2\right)\left(\frac{d^2}{dx^2}+4^2\right)\ldots\left\{\frac{d^2}{dx^2}+(2r)^2\right\}\phi x$;

theorems which are generally true, whether the development of $\phi(x)$ be free from negative indices or not.

To enter upon the general illustration of the above theorems, would extend this paper beyond its proper limits: two or three examples must therefore suffice. For this purpose I select the last two theorems, (25) and (26).

In (26), let $\phi x = \cos mx$; then, by a known theorem, $\psi\left(\frac{d^2}{dx^2}\right)\cos mx = \psi(-m^2)\cos mx$, whence

$$\int_0^\pi (\sin x)^{2r}\cos mx = \frac{\Gamma(2r+1)}{(2^2-m^2)(4^2-m^2)..\{(2r)^2-m^2\}}\,\frac{\sin m\pi}{m}.$$

In (26) again, let $r = 2$, then

$$\int_0^\pi (\sin x)^2 \left(\frac{d^2}{dx^2} + 4\right) \phi x \, dx = 2 \int_0^\pi \phi(x)\, dx. \ldots (27).$$

Let $\phi(x) = \dfrac{1}{1+x}$, whence $\displaystyle\int_0^\pi dx\, \phi(x) = \log(1+\pi)$, and effecting the differentiations in the first member, we find

$$\int_0^\pi dx\, (\sin x)^2 \, \frac{2 + 2x + x^2}{(1+x)^3} = \log(1+\pi);$$

and similarly, from (25),

$$\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} dx\, (\cos x)^2 \, \frac{2 + 2x + x^2}{(1+x)^3} = \log\left(\frac{\pi^2}{4} - 1\right).$$

Let $\phi(x) = x^m$, then by (27), m being $> (-1)$

$$\int_0^\pi dx \sin x^2 \left\{x^m + \frac{m(m-1)\, x^{m-2}}{4}\right\} = \frac{\pi^{m+1}}{m+1}.$$

The above results I have verified independently.

VI.—ON THE MOTION OF ROTATION OF A SOLID BODY.

By Arthur Cayley, B.A. Fellow of Trinity College.

In the fifth volume of Liouville's Journal, in a paper "Des lois géometriques qui regissent les deplacemens d'un systeme solide," M. Olinde Rodrigues has given some very elegant formulæ for determining the position of two sets of rectangular axes with respect to each other, employing rational functions of three quantities only. The principal object of the present paper is to apply these to the problem of the rotation of a solid body; but I shall first demonstrate the formulæ in question, and some others connected with the same subject which may be useful on other occasions.

Let Ax, Ay, Az; $Ax_{,}$, $Ay_{,}$, $Az_{,}$, be any two sets of rectangular axes passing through the point A, x, y, z; $x_{,}, y_{,}, z_{,}$ being taken for the points where these lines intersect the spherical surface described round the centre A with radius unity. Join $xx_{,}$, $yy_{,}$, $zz_{,}$, by arcs of great circles, and through the central points of these describe great circles cutting them at right angles. These are easily seen to intersect in a certain point P. Let $Px = f$, $Py = g$, $Pz = h$; then also $Px_{,} = f$, $Py_{,} = g$, $Pz_{,} = h$. And $\angle xPx_{,} = \angle yPy_{,} = \angle zPz_{,} = \theta$ suppose, θ being measured from xP towards yP, yP towards zP, or

$_zP$ towards xP. The cosines of f, g, h, are of course connected by the equation

$$\cos^2 f + \cos^2 g + \cos^2 h = 1.$$

Let α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$, represent the cosines of x_x, y_x, z_x; x_y, y_y, z_y; x_z, y_z, z_z: these quantities are to be determined as functions of f, g, h, θ.

Suppose for a moment,

$$\angle yPz = \mathrm{x}, \quad \angle zPx = \mathrm{y}, \quad \angle xPy = \mathrm{z}.$$

Then
$$\begin{aligned}
\alpha &= \cos^2 f + \sin^2 f \cos\theta,\\
\alpha' &= \cos f \cos g + \sin f \sin g \cos\{(\mathrm{z} - \theta)\},\\
\alpha'' &= \cos f \cos h + \sin f \sin h \cos\{(\mathrm{y} + \theta)\}.\\
\beta &= \cos g \cos f + \sin g \sin f \cos\{(\mathrm{z} + \theta)\},\\
\beta' &= \cos^2 g + \sin^2 g \cos\theta,\\
\beta'' &= \cos g \cos h + \sin g \sin h \cos\{(\mathrm{x} - \theta)\}.\\
\gamma &= \cos h \cos f + \sin h \sin f \cos\{(\mathrm{y} - \theta)\},\\
\gamma' &= \cos h \cos g + \sin h \sin g \cos\{(\mathrm{x} + \theta)\},\\
\gamma'' &= \cos^2 h + \sin^2 h \cos\theta.
\end{aligned}$$

Also
$$\begin{aligned}
\sin g \sin h \cos \mathrm{x} &= -\cos g \cos h,\\
\sin h \sin f \cos \mathrm{y} &= -\cos h \cos f,\\
\sin f \sin g \cos \mathrm{z} &= -\cos f \cos g;
\end{aligned}$$

and
$$\begin{aligned}
\sin g \sin h \sin \mathrm{x} &= \cos f,\\
\sin h \sin f \sin \mathrm{y} &= \cos g,\\
\sin f \sin g \sin \mathrm{z} &= \cos h.
\end{aligned}$$

Substituting,
$$\begin{aligned}
\alpha &= \cos^2 f + \sin^2 f \cos\theta,\\
\alpha' &= \cos f \cos g\,(1 - \cos\theta) + \cos h \sin\theta,\\
\alpha'' &= \cos f \cos h\,(1 - \cos\theta) - \cos g \sin\theta.\\
\beta &= \cos g \cos f\,(1 - \cos\theta) - \cos h \sin\theta,\\
\beta' &= \cos^2 g + \sin^2 g \cos\theta,\\
\beta'' &= \cos g \cos h\,(1 - \cos\theta) + \cos f \sin\theta.\\
\gamma &= \cos h \cos f\,(1 - \cos\theta) + \cos g \sin\theta,\\
\gamma' &= \cos h \cos g\,(1 - \cos\theta) - \cos f \sin\theta,\\
\gamma'' &= \cos^2 h + \sin^2 h \cos\theta.
\end{aligned}$$

Assume $\lambda = \tan\frac{1}{2}\theta \cos f$, $\mu = \tan\frac{1}{2}\theta\cos g$, $\nu = \tan\frac{1}{2}\theta \cos h$, and $\sec^2\frac{1}{2}\theta = 1 + \lambda^2 + \mu^2 + \nu^2 = \kappa$. Then

$$\begin{array}{lll}
\kappa\alpha = 1 + \lambda^2 - \mu^2 - \nu^2, & \kappa\alpha' = 2(\lambda\mu + \nu), & \kappa\alpha'' = 2(\nu\lambda - \mu),\\
\kappa\beta = 2(\lambda\mu - \nu), & \kappa\beta' = 1 + \mu^2 - \nu^2 - \lambda^2, & \kappa\beta'' = 2(\mu\nu + \lambda);\\
\kappa\gamma = 2(\nu\lambda + \mu), & \kappa\gamma' = 2(\mu\nu - \lambda), & \kappa\gamma'' = 1 + \nu^2 - \lambda^2 - \mu^2;
\end{array}$$

which are the formulæ required, differing only from those in Liouville, by having λ, μ, ν, instead of $\frac{1}{2}m$, $\frac{1}{2}n$, $\frac{1}{2}p$; and $\alpha, \alpha', \alpha''$; β, β', β''; $\gamma, \gamma', \gamma''$, instead of a, b, c; a', b', c'; a'', b'', c''. It is to be remarked, that β', β'', β; $\gamma'', \gamma, \gamma'$, are deduced from $\alpha, \alpha', \alpha''$, by writing μ, ν, λ; ν, λ, μ, for λ, μ, ν.

Let $1 + \alpha + \beta' + \gamma'' = \upsilon$;

$$\kappa\upsilon = 4,$$

$$\lambda\upsilon = \beta'' - \gamma', \qquad \mu\upsilon = \gamma - \alpha'', \qquad \kappa\upsilon = \alpha' - \beta,$$

$$\lambda^2\upsilon = 1 + \alpha - \beta' - \gamma'', \quad \mu^2\upsilon = 1 - \alpha + \beta' - \gamma'', \quad \nu^2\upsilon = 1 - \alpha - \beta' - \gamma''.$$

Suppose that Ax, Ay, Az, are referred to axes AX, AY, AZ, by the quantities l, m, n, k, analogous to $\lambda, \mu, \nu, \kappa$, these latter axes being referred to $Ax_{,}, Ay_{,}, Az_{,}$, by the quantities $l_{,}, m_{,}, n_{,}, k_{,}$.

Let a, b, c; a', b', c'; a'', b'', c''; $a_{,}, b_{,}, c_{,}$; $a_{,}', b_{,}', c_{,}'$; $a_{,}'', b_{,}'', c_{,}''$, denote the quantities analogous to α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$. Then we have, by spherical trigonometry, the formulæ

$$\alpha = aa_{,} + ba_{,}' + ca_{,}'', \qquad \beta = ab_{,} + bb_{,}' + cb_{,}'', \qquad \gamma = ac_{,} + bc_{,}' + cc_{,}'';$$

$$\alpha' = a'a_{,} + b'a_{,}' + c'a_{,}'', \qquad \beta' = a'b_{,} + b'b_{,}' + c'b_{,}'', \qquad \gamma' = a'c_{,} + b'c_{,}' + c'c_{,}'';$$

$$\alpha'' = a''a_{,} + b''a_{,}' + c''a_{,}'', \qquad \beta'' = a''b_{,} + b''b_{,}' + c''b_{,}'', \qquad \gamma'' = a''c_{,} + b''c_{,}' + c''c_{,}''.$$

Then expressing a, b, c; a', b', c'; a'', b'', c''; $a_{,}, b_{,}, c_{,}$; $a_{,}', b_{,}', c_{,}'$; $a_{,}'', b_{,}'', c_{,}''$, in terms of l, m, n; $l_{,}, m_{,}, n_{,}$, after some reductions we arrive at

$$kk_{,}\upsilon = 4(1 - ll_{,} - mm_{,} - nn_{,})^2 = 4\Pi^2 \text{ suppose,}$$

$$kk_{,}(\beta'' - \gamma) = 4(l + l_{,} + n_{,}m - nm_{,})\,\Pi,$$

$$kk_{,}(\gamma - \alpha') = 4(m + m_{,} + l_{,}m - lm_{,})\,\Pi,$$

$$kk_{,}(\alpha' - \beta'') = 4(n + n_{,} + m_{,}n - mn_{,})\,\Pi.$$

And hence

$$\Pi = 1 - ll_{,} - mm_{,} - nn_{,}, \qquad \Pi\lambda = l + l_{,} + n_{,}m - nm_{,},$$

$$\Pi\mu = m + m_{,} + l_{,}m - lm_{,}, \qquad \Pi\nu = n + n_{,} + m_{,}n - mn_{,},$$

which are formulæ of considerable elegance for exhibiting the combined effect of successive displacements of the axes. The following analogous ones are readily obtained:

$$P = 1 + \lambda l + \mu m + \nu n, \qquad Pl_{,} = \lambda - l - \nu m + \mu n,$$

$$Pm_{,} = \mu - m - \lambda n + \nu l, \qquad Pn_{,} = \nu - n - \mu l + \lambda m:$$

and again,

$$P_{,} = 1 + \lambda l_{,} + \mu m_{,} + \nu n_{,}, \qquad P_{,}l = \lambda - l_{,} + \nu m_{,} - \mu n_{,},$$

$$P_{,}m = \mu - m_{,} + \lambda n_{,} - \nu l_{,}, \qquad P_{,}n = \nu - n_{,} + \mu l_{,} - \lambda m_{,}.$$

These formulæ will be found useful in the integration of the equations of rotation of a solid body.

Next it is required to express the quantities $p, q, r,$ in terms of $\lambda, \mu, \nu,$ where as usual

$$p = \gamma \frac{d\beta}{dt} + \gamma' \frac{d\beta'}{dt} + \gamma'' \frac{d\beta''}{dt},$$

$$q = \alpha \frac{d\gamma}{dt} + \alpha' \frac{d\gamma'}{dt} + \alpha'' \frac{d\gamma''}{dt},$$

$$r = \beta \frac{d\alpha}{dt} + \beta' \frac{d\alpha'}{dt} + \beta'' \frac{d\alpha''}{dt}.$$

Differentiating the values of $\beta\kappa$, $\beta'\kappa$, $\beta''\kappa$, multiplying by $\gamma, \gamma', \gamma''$, and adding,

$$\kappa p = 2\lambda'(\gamma\mu - \gamma'\lambda + \gamma'') + 2\mu'(\gamma\lambda - \gamma'\mu + \gamma''\nu) + 2\nu'(-\gamma - \gamma'\nu + \gamma''\mu),$$

where λ', μ', ν', denote $\frac{d\lambda}{dt}, \frac{d\mu}{dt}, \frac{d\nu}{dt}$. Reducing, we have

$$\kappa p = 2(\lambda' + \nu\mu' - \nu'\mu):$$

from which it is easy to derive the system

$$\kappa p = 2(\lambda' + \nu\mu' - \nu'\mu),$$
$$\kappa q = 2(-\nu\lambda' + \mu' + \nu'\lambda),$$
$$\kappa r = 2(\mu\lambda' - \mu'\lambda + \nu');$$

or, determining λ', μ', ν', from these equations, the equivalent system

$$2\lambda' = (1 + \lambda^2)p + (\lambda\mu - \nu)q + (\nu\lambda + \mu)r,$$
$$2\mu' = (\lambda\mu + \nu)p + (1 + \mu^2)q + (\mu\nu - \lambda)r,$$
$$2\nu' = (\nu\lambda - \mu)p + (\mu\nu + \lambda)q + (1 + \nu^2)r.$$

The following equation also is immediately obtained,

$$\kappa' = \kappa(\lambda p + \mu q + \nu r).$$

The subsequent part of the problem requires the knowledge of the differential coefficients of $p, q, r,$ with respect to $\lambda, \mu, \nu; \lambda', \mu', \nu'$. It will be sufficient to write down the six.

$$\kappa \frac{dp}{d\lambda'} = 2, \qquad \kappa \frac{dp}{d\lambda} + 2p\lambda = 0,$$

$$\kappa \frac{dq}{d\lambda'} = -2\nu, \qquad \kappa \frac{dq}{d\lambda} + 2q\lambda = 2\nu',$$

$$\kappa \frac{dr}{d\lambda'} = 2\mu, \qquad \kappa \frac{dr}{d\lambda} + 2r\lambda = -2\mu',$$

from which the others are immediately obtained.

Suppose now a solid body acted on by any forces, and revolving round a fixed point. The equations of motion are

$$\frac{d}{dt}\cdot\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=\frac{dV}{d\lambda},$$

$$\frac{d}{dt}\cdot\frac{dT}{d\mu'}-\frac{dT}{d\mu}=\frac{dV}{d\mu},$$

$$\frac{d}{dt}\cdot\frac{dT}{d\nu'}-\frac{dT}{d\nu}=\frac{dV}{d\nu}.$$

Where $T=\frac{1}{2}(Ap^2+Bq^2+Cr^2)$,

$$V=\Sigma\left[\int(Xdx+Ydy+Zdz)\right]dm.$$

Or if $Xdx+Ydy+Zdz$ is not an exact differential, $\frac{dV}{d\lambda}, \frac{dV}{d\mu}, \frac{dV}{d\nu}$, are independent symbols standing for

$$\Sigma\left(X\frac{dx}{d\lambda}+Y\frac{dy}{d\lambda}+Z\frac{dz}{d\lambda}\right)dm, \ldots.$$

Vide *Mécanique Analytique*, Avertissement, t. I. p. 4. Only in this latter case V stands for the disturbing function, the principal forces vanishing.

Now, considering the first of the above equations

$$\frac{dT}{d\lambda'}=Ap\frac{dp}{d\lambda}+Bq\frac{dq}{d\lambda}+Cr\frac{dr}{d\lambda}$$

$$=\frac{2}{\kappa}(Ap-\nu Bq+\mu Cr).$$

Whence, writing p', q', r', κ', for $\frac{dp}{dt}, \frac{dq}{dt}, \frac{dr}{dt}, \frac{d\kappa}{dt}$,

$$\frac{d}{dt}\left(\frac{dT}{d\lambda'}\right)=\frac{2}{\kappa}(Ap'-\nu Bq'+\mu Cr')-\frac{2}{\kappa}Bq\nu'$$

$$+\frac{2}{\kappa}Cr\mu'-\frac{2\kappa'}{\kappa^2}(Ap-\nu Bq+\mu Cr).$$

Also $\frac{dT}{d\lambda}=Ap\frac{dp}{d\lambda}+Bq\frac{dq}{d\lambda}+Cr\frac{dr}{d\lambda}$

$$=-\frac{2\lambda}{\kappa}(Ap^2+Bq^2+Cr^2)+\frac{2}{\kappa}Bq\nu'-\frac{2}{\kappa}Cr\mu'.$$

And hence

$$\frac{1}{2}\left(\frac{d}{dt}\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}\right)=\frac{1}{\kappa}(Ap'-\nu Bq'+\mu Cr')-\frac{2}{\kappa}Bq\nu'+\frac{2}{\kappa}Cr\mu'$$

$$+\frac{\lambda}{\kappa}(Ap^2+Bq^2+Cr^2)-\frac{\kappa'}{\kappa^2}(Ap-\nu Bq+\mu Cr).$$

Substituting for $\lambda', \mu', \nu', \kappa'$, after all reductions,

$$\frac{1}{2}\left(\frac{d}{dt}\cdot\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}\right)=\frac{1}{\kappa}[\{Ap'+(C-B)qr\}-\nu\{Bq'+(A-C)rp\}$$

$$+\mu\{Cr'+(B-A)pq\}].$$

And, forming the analogous quantities in μ, ν, and substituting in the equations of motion, these become

$$Ap'+(C-B)qr-\nu\{Bq'+(A-C)rp\}+\mu\{Cr'+(B-A)pq\}=\tfrac{1}{2}\kappa\frac{dV}{d\lambda},$$

$$\nu\{Ap'+(C-B)qr\}+\{Bq'+(A-C)rp\}-\lambda\{Cr'+(B-A)pq\}=\tfrac{1}{2}\kappa\frac{dV}{d\mu},$$

$$\mu\{Ap'+(C-B)qr\}+\lambda\{Bq'+(A-C)rp\}+\{Cr'+(B-A)pq\}=\tfrac{1}{2}\kappa\frac{dV}{d\nu}.$$

Or eliminating, and replacing p', q', r', by $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$,

$$A\frac{dp}{dt}+(C-B)qr=\frac{1}{2}\left\{(1+\lambda^2)\frac{dV}{d\lambda}+(\lambda\mu+\nu)\frac{dV}{d\mu}+(\nu\lambda-\mu)\frac{dV}{d\nu}\right\},$$

$$B\frac{dq}{dt}+(A-C)rp=\frac{1}{2}\left\{(\lambda\mu-\nu)\frac{dV}{d\lambda}+(1+\mu^2)\frac{dV}{d\mu}+(\mu\nu+\lambda)\frac{dV}{d\nu}\right\},$$

$$C\frac{dr}{dt}+(B-A)pq=\frac{1}{2}\left\{(\nu\lambda+\mu)\frac{dV}{d\lambda}+(\mu\nu-\lambda)\frac{dV}{d\mu}+(1+\nu^2)\frac{dV}{d\nu}\right\};$$

to which are to be joined

$$\kappa p = 2\left(\frac{d\lambda}{dt}+\nu\frac{d\mu}{dt}-\mu\frac{d\nu}{dt}\right),$$

$$\kappa q = 2\left(-\nu\frac{d\lambda}{dt}+\frac{d\mu}{dt}+\lambda\frac{d\nu}{dt}\right),$$

$$\kappa r = 2\left(\mu\frac{d\lambda}{dt}-\lambda\frac{d\mu}{dt}+\frac{d\nu}{dt}\right);$$

where it will be recollected

$$\kappa = 1+\lambda^2+\mu^2+\nu^2;$$

and on the integration of these six equations depends the complete determination of the motion.

If we neglect the terms depending on V, the first three equations may be integrated in the form

$$p^2=p_1^{\,2}-\frac{C-B}{A}\phi.\quad q^2=q_1^{\,2}-\frac{A-C}{B}\phi.\quad r^2=r_1^{\,2}-\frac{B-A}{C}\phi.$$

$$2t=\int\frac{d\phi}{\left\{\left(p_1^{\,2}-\frac{C-B}{A}\phi\right)\left(q_1^{\,2}-\frac{A-C}{B}\phi\right)\left(r_1^{\,2}-\frac{B-A}{C}\phi\right)\right\}^{\frac{1}{2}}}.$$

And considering p, q, r as functions of ϕ, given by these equations, the three latter ones take the form

$$\frac{\kappa}{4qr} = \frac{d\lambda}{d\phi} + \nu\frac{d\mu}{d\phi} - \mu\frac{d\nu}{d\phi},$$

$$\frac{\kappa}{4rp} = -\nu\frac{d\lambda}{d\phi} + \frac{d\mu}{d\phi} + \lambda\frac{d\nu}{d\phi},$$

$$\frac{\kappa}{4pq} = \mu\frac{d\lambda}{d\phi} - \lambda\frac{d\mu}{d\phi} + \frac{d\nu}{d\phi};$$

of which, as is well known, the equations following, equivalent to two independent equations, are integrals.

$$\kappa g = Ap(1+\lambda^2-\mu^2-\nu^2) + 2Bq(\lambda\mu-\nu) + 2Cr(\nu\lambda+\mu),$$

$$\kappa g' = 2Ap(\lambda\mu+\nu) + Bq(1+\mu^2-\lambda^2-\nu^2) + 2Cr(\mu\nu-\lambda),$$

$$\kappa g'' = 2Ap(\nu\lambda-\mu) + 2Bq(\mu\nu+\lambda) + Cr(1+\nu^2-\lambda^2-\mu^2);$$

where g, g', g'', are arbitrary constants satisfying

$$g^2 + g'^2 + g''^2 = A^2p_1^{\,2} + B^2q_1^{\,2} + C^2r_1^{\,2}.$$

To obtain another integral, it is apparently necessary, as in the ordinary theory, to revert to the consideration of the invariable plane. Suppose $g' = 0$, $g'' = 0$.

Then $\quad g'' = \sqrt{(A^2p_1^{\,2} + B^2q_1^{\,2} + C^2r_1^{\,2})} = k$ suppose.

We easily obtain, where $\lambda_0, \mu_0, \nu_0, \kappa_0$ are written for $\lambda, \mu, \nu, \kappa$, to denote this particular supposition,

$$\kappa_0 Ap = 2(\nu_0\lambda_0 - \mu_0)k,$$

$$\kappa_0 Bq = 2(\mu_0\nu_0 + \lambda_0)k,$$

$$\kappa_0 Cr = (1 + \nu_0^{\,2} - \lambda_0^{\,2} - \mu_0^{\,2})k;$$

whence, and from $\quad \kappa_0 = 1 + \lambda_0^{\,2} + \mu_0^{\,2} + \nu_0^{\,2}$,

$$\kappa_0 Cr = (2 + 2\nu_0^{\,2} - \kappa_0)k,$$

$$\kappa_0 = \frac{(2+2\nu_0^{\,2})k}{k+Cr}.$$

And therefore

$$\nu_0\lambda_0 - \mu_0 = \frac{(1+\nu_0^{\,2}).Ap}{k+Cr}, \quad \mu_0\nu_0 + \lambda_0 = \frac{(1+\nu_0^{\,2}).Bq}{k+Cr}.$$

Hence, writing $h = Ap_1^{\,2} + Bq_1^{\,2} + Cr_1^{\,2}$, the equation

$$\frac{d\nu_0}{d\phi} = \frac{1}{4pqr}\left\{(\nu_0\lambda_0 - \mu_0)p + (\mu_0\nu_0 + \lambda_0)q + (1+\nu^2)r\right\},$$

reduces itself to

$$\frac{4}{1+\nu_0^{\,2}}\cdot\frac{d\nu_0}{d\phi} = \frac{h+kr}{(k+Cr)pqr},$$

$$\text{or} \qquad 4\tan^{-1}\nu_0 = \int\frac{(h+kr).d\phi}{(k+Cr)pqr}.$$

The integral taking rather a simpler form if p, q, ϕ be considered functions of r, and becoming

$$2\tan^{-1}\nu_0 = \int \frac{h+kr}{k+Cr} \frac{C\sqrt{(AB)}\,.\,dr}{\sqrt{[\{k^2-Bh+(B-C)Cr^2\}\{Ah-k^2+(C-A)Cr^2\}]}};$$

and (ν_0) being determined, λ_0, μ_0, are given by the equations

$$\lambda_0 = \frac{\nu_0 Ap + Bq}{k + Cr}, \quad \mu_0 = \frac{\nu_0 Bq - Ap}{k + Cr}.$$

Hence l, m, n, denoting arbitrary constants, the general values of λ, μ, ν, are given by the equations

$$P_0 = 1 - l\lambda_0 - m\mu_0 - n\nu_0,$$
$$P_0\lambda = l + \lambda_0 + \nu_0 m - \mu_0 n,$$
$$P_0\mu = m + \mu_0 + \lambda_0 n - \nu_0 l,$$
$$P_0\nu = n + \nu_0 + \mu_0 l - \lambda_0 m.$$

In a following paper I propose to develope the formulæ for the variations of the arbitrary constants p_1, q_1, r_1, l, m, n, when the terms involving V are taken into account.

Note.—It may be as well to verify independently the analytical conclusion immediately deducible from the preceding formulæ, viz. if λ, μ, ν, be given by the differential equations,

$$\kappa p = \frac{d\lambda}{dt} + \nu\frac{d\mu}{dt} - \mu\frac{d\nu}{dt},$$

$$\kappa q = -\nu\frac{d\lambda}{dt} + \frac{d\mu}{dt} + \lambda\frac{d\nu}{dt},$$

$$\kappa r = \mu\frac{d\lambda}{dt} - \lambda\frac{d\mu}{dt} + \frac{d\nu}{dt},$$

where $\kappa = 1 + \lambda^2 + \mu^2 + \nu^2$, and p, q, r, are any functions of t. Then if λ_0, μ_0, ν_0, be particular values of λ, μ, ν, and l, m, n, arbitrary constants, the general integrals are given by the system

$$P_0 = 1 - l\lambda_0 - m\mu_0 - n\nu_0,$$
$$P_0\lambda = l + \lambda_0 + \nu_0 m - \mu_0 n,$$
$$P_0\mu = m + \mu_0 + \lambda_0 n - \nu_0 l,$$
$$P_0\nu = n + \nu_0 + \mu_0 l - \lambda_0 m.$$

Assuming these equations, we deduce the equivalent system,

$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, l = \lambda - \lambda_0 + \nu_0\mu - \nu\mu_0,$$
$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, m = \mu - \mu_0 + \lambda_0\nu - \lambda\nu_0,$$
$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, n = \nu - \nu_0 + \mu_0\lambda - \mu\lambda_0.$$

Differentiate the first of these and eliminate l, the result takes the form

$$\begin{aligned}
&-(\mu_0^2 + \nu_0^2)(\lambda' + \nu\mu' - \nu'\mu) - (\nu_0 - \lambda_0\mu_0)(-\nu\lambda' + \mu' + \lambda\nu') \\
&\qquad\qquad + (\mu_0 + \lambda_0\nu_0)(\mu\lambda' - \lambda\mu' + \nu') + \kappa_0\lambda', \\
&+(\mu^2 + \nu^2)(\lambda_0' + \nu_0\mu_0' - \nu_0'\mu_0) + (\nu - \lambda\mu)(-\nu_0\lambda_0' + \mu_0' + \lambda_0\nu_0') \\
&\qquad\qquad - (\mu + \lambda\nu)(\mu_0\lambda_0' - \lambda_0\mu_0' + \nu_0') - \kappa\lambda_0' = 0,
\end{aligned}$$

Where λ', &c. denote $\dfrac{d\lambda}{dt}$, &c.

$$\kappa_0 = 1 + \lambda_0^2 + \mu_0^2 + \nu_0^2.$$

Reducing by the differential equations in λ, μ, ν; λ_0, μ_0, ν_0, this becomes

$$\kappa_0 \{\lambda' + \tfrac{1}{2}p(\mu^2 + \nu^2) + \tfrac{1}{2}q(\nu - \lambda\mu) - \tfrac{1}{2}r(\mu + \lambda\nu)\}$$
$$- \kappa \{\lambda_0' + \tfrac{1}{2}p(\mu_0^2 + \nu_0^2) + \tfrac{1}{2}q(\nu_0 - \lambda_0\mu_0) - \tfrac{1}{2}r(\mu_0 + \lambda\nu_0)\} = 0;$$

or substituting for λ', λ_0', this reduces itself to the identical equation
$$\tfrac{1}{2}p(\kappa_0\kappa - \kappa\kappa_0) = 0:$$
and similarly may the remaining equations be verified.

VII.—SOLUTION OF A PROBLEM IN ANALYTICAL GEOMETRY.

THE following problem is given in *Grunert's Archiv der Mathematik und Physik*, I. 136, and as it has not yet found a place in any treatise on Analytical Geometry, it may be new to most of our readers. The demonstration differs from that of Grunert in the use of the symmetrical equations to the straight line; in other respects the method is the same.

Let the equations to the four given lines be

$$\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \ldots.(1), \quad \frac{x-a_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \ldots.(2),$$
$$\frac{x-a_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \ldots.(3), \quad \frac{x-a_3}{l_3} = \frac{y-\beta_3}{m_3} = \frac{z-\gamma_3}{n_3} \ldots.(4),$$

and assume the required equations to be

$$\frac{x-x'}{\lambda} = \frac{y-y'}{\mu} = \frac{z-z'}{\nu} \ldots\ldots(5).$$

Since x', y', z', are only limited to be the co-ordinates of some point in the line (5), we may assume them to be those of the intersection of (1) and (5), which gives us the equations

$$\frac{x'-a}{l} = \frac{y'-\beta}{m} = \frac{z'-\gamma}{n} = r \quad \text{suppose}\ldots\ldots(6).$$

But the conditions for the intersection of (5) and (2), of (5) and (3), and of (4) and (3) are

$$\left.\begin{aligned} L_1(x'-a_1) + M_1(y'-\beta_1) + N_1(z'-\gamma_1) = 0 \\ L_2(x'-a_2) + M_2(y'-\beta_2) + N_2(z'-\gamma_2) = 0 \\ L_3(x'-a_3) + M_3(y'-\beta_3) + N_3(z'-\gamma_3) = 0 \end{aligned}\right\} \ldots.(7),$$

if we put for shortness

$$L_1 = m_1\nu - n_1\mu, \quad M_1 = n_1\lambda - l_1\nu, \quad N_1 = l_1\mu - m_1\lambda,$$

and similarly for the other quantities.

But $x' - a' = x' - a - (a_1 - a)$ and similarly for the others; hence, substituting in (7) the value of $x' - a$, $y' - \beta$, $z' - \gamma$, derived from (6) we have three equations of which the first is

$$L_1\{lr - (a_1 - a)\} + M_1\{mr - (\beta_1 - \beta)\} + N_1\{nr - (\gamma_1 - \gamma)\} = 0.$$

For L_1, M_1, N_1, substitute these values and arrange the expressions in terms of λ, μ, ν; and putting for shortness

$$A_1 = n_1m - m_1n, \quad B_1l = l_1n - n_1l, \quad C_1 = m_1l - l_1m$$

$$a_1 = m_1(\gamma_1 - \gamma) - n_1(\beta_1 - \beta), \quad b_1 = n_1(a_1 - a) - l_1(\gamma_1 - \gamma),$$

$$c_1 = l_1(\beta_1 - \beta) - m_1(a_1 - a),$$

we obtain the equation

$$(A_1r + a_1)\lambda + (B_1r + b_1)\mu + (C_1r + c_1)\nu = 0 \ \ldots\ldots (8).$$

In like manner, from the other equations of (7) we have

$$(A_2r + a_2)\lambda + (B_2r + b_2)\mu + (C_2r + c_2)\nu = 0 \ \ldots\ldots (9),$$

$$(A_3r + a_3)\lambda + (B_3r + b_3)\mu + (C_3r + c_3)\nu = 0 \ \ldots\ldots (10).$$

These three equations, along with the condition

$$\lambda^2 + \mu^2 + \nu^2 = 1,$$

are sufficient to determine λ, μ, ν and r, and from r the values of x', y', z' are found by (6). The best way of proceeding is first to eliminate λ, μ, ν from (8), (9), (10) by cross multiplication, when we obtain an apparent cubic of the form

$$\begin{aligned}&(A_1r + a_1)\{(B_2r + b_2)(C_3r + c_3) - (B_3r + b_3)(C_2r + c_2)\} + \\ &(A_2r + a_2)\{(B_3r + b_3)(C_1r + c_1) - (B_1r + b_1)(C_3r + c_3)\} + \\ &(A_3r + a_3)\{(B_1r + b_1)(C_2r + c_2) - (B_2r + b_2)(C_3r + c_1)\} = 0.\end{aligned}$$

But if the coefficient of r^3 be examined it will be found to be identically equal to zero, so that the cubic is reduced to a quadratic of the form

$$Pr^2 + Qr + R = 0.$$

This gives two values of r: if they are both possible, there are two corresponding sets of values for λ, μ, ν, x', y', z', or two lines which satisfy the conditions of the problem: if the two values of r are impossible, the problem is impossible. When the values of r are known, they are to be substituted successively in the equations (8), (9), (10), from which λ, μ, ν may be determined in the ordinary way; but the expressions are so long and complicated that it is useless to write them down here.

VIII.—ON THE KNIGHT'S MOVE AT CHESS.

By R. Moon, M.A., Fellow of Queens' College.

It is some time since Dr. Roget published in the *Philosophical Magazine* his solution of the problem of the "Knight at Chess," when the initial and terminal squares are given. It

so happened that, a short time previous to the appearance of Dr. Roget's paper, my attention was directed to the subject by a friend, to whom it had been suggested by the perusal of a German work; and the result of my attempts at that time was the discovery of a method of solving the problem, (freed indeed from Dr. Roget's restriction as to the terminal square,) not only in the case of the common board of eight squares to a side, but also when the number of squares to a side is twelve, sixteen, or any multiple of four. I have more recently found, that the same method is applicable whatever be the number of sides, provided they exceed four; and with this further reservation, that if the number of squares be odd, and the central square be black when the initial square be white, *one* square must be allowed to be left uncovered. In fact, it is easy to see, *à priori*, that this last must be granted as a postulate; since, in the case supposed, the number of black squares in the board exceeds the number of white by one; and, by the conditions of the problem, a white and a black square must be covered alternately.

d	*c*	*b*	*a*	*d*	*c*	*b*	*a*
b	*a*	*d*	*c*	*b*	*a*	*d*	*c*
c	*d*	*D*	*C*	*B*	*A*	*a*	*b*
a	*b*	*B*	*A*	*D*	*C*	*c*	*d*
d	*c*	*C*	*D*	*A*	*B*	*b*	*a*
b	*a*	*A*	*B*	*C*	*D*	*d*	*c*
c	*d*	*a*	*b*	*c*	*d*	*a*	*b*
a	*b*	*c*	*d*	*a*	*b*	*c*	*d*

Take the case of the common board of eight squares to a side, and consider it as made up of a central square of four places to a side and the annulus about it.

If the Knight be placed on any square of the annulus, and be confined to move in it, it will return to the place from which it set out, after describing what we shall call a circuit. Thus, in the figure, the places marked with the letter *a* constitute a circuit. If the piece be placed on any square of the annulus not included in the first circuit, it may be made to describe a second distinct from the first; and the whole

annulus may be divided into four such circuits, which are respectively made up of the places marked with the letters a, b, c, d, respectively. The central square may be divided into four similar circuits. Also, if we have a board of twelve squares to a side, we may consider it as made up of a central square of eight places to a side, (which again may be subdivided in the manner above indicated), and an external annulus, which may be divided into four circuits; and similarly if we have a board of sixteen or twenty squares to a side.

Now it will be found, if we confine ourselves to one particular annulus, that it will in general be impossible to pass from one circuit to another. But if after describing a circuit in any annulus, or in the central square (which we will call principal divisions), we wish to pass to any other principal division, it will be found that, when we are moving *from* the centre, we may always pass from a particular letter in one division to any of the three other letters in the other division: thus, in the figure, we may always pass from A to b, d, and c, and so on. On the other hand, if we are moving *towards* the centre, in passing from one division to another the change may always be effected from any particular letter in the one to its *opposite* in the other, as from a to D, or from b to C, &c. Hence it is easy to write down the order in which the circuits should succeed each other for the common chess-board. Thus, if the first place be an a, the cycle will be

$$a\ D\ b\ C\ d\ A\ c\ B,$$

$$\text{or}\quad a\ D\ c\ B\ d\ A\ b\ C.$$

It is not meant that these are the only cycles which would succeed, but they are such as cannot fail, and the method of obtaining them is obvious.

When the board consists of twelve or more squares, some additional considerations are requisite. In this case, in fact, it is obvious that, if the initial place be in the outer annulus, and after describing one circuit in that we proceed to describe another in the middle division, and a third in the central square, and so move backwards and forwards from the outermost to the innermost division, and *vice versâ*, describing only one circuit in each division each time it is entered, we shall have exhausted the middle division before we have exhausted the two others; at the same time that it is impossible to proceed from the inner to the outer division *per saltum*, or without having recourse to the intermediate one. To obviate this difficulty, it must be observed in the first place, that in the central square it is always possible to move from B or C to A or D; so that if we arrive in the central square on a B or a C, we may describe two circuits at once,

and it will be found that this may *always* be effected. In the next place, it will be found, that in the external annulus we may pass from the circuit of *b* or *c* to that of *a* or *d*, or *vice versâ*, and thus describe two circuits at once, provided that, when we have concluded the first circuit, we are sufficiently near to the corners of the board; and this may always be contrived as we move *down* from the centre.

Hence, when the number of squares to a side is greater than eight, the rule is, to move in the first place up to the centre, and to continue moving backwards and forwards, taking care to cover two circuits of the innermost and outermost divisions whenever those divisions are entered.

Thus far as to the case when the number of squares to a side is a multiple of four. If the board be one of six squares to a side, it may be divided into a central square, containing four places in all, and an annulus which may be divided into four circuits. A board of ten places to a side may be divided into a central square of six places to a side, and an external annulus likewise divisible into four circuits; and similarly when the number of squares to a side is eighteen, twenty-two, &c. In all these cases, when the number to a side is greater than six, the central square of six must be treated in all respects as the central square of four to a side, when the number of places in the side of the board is a multiple of four, that is, when once entered it must be exhausted one half. In other respects the method is the same in the two cases.

If the number of squares in the board be odd, the same principle of division obtains. We shall still have a central square, which will have either five or seven places to a side, as the case may be, and a series of annuli divisible into circuits. It will be found, however, that in this case each annulus will consist of only two circuits; by reason of which the process is much simplified.

The limits necessarily prescribed in a publication of this kind, do not admit of my detailing the method of exhausting the squares of five, six, and seven places to a side; but after what has been said, it is probable that these will not present much difficulty. For the same reason I shall not attempt to point out the order of the circuits; to do which, so as to meet every case, would lead us to great length. I shall merely remark in conclusion, that it would not be difficult to show how Dr. Roget's restriction as to the final square might be complied with, if it were worth while to do so.

[A very convenient practical solution of the general problem on the ordinary board, is given in a work, "*Indian Reminiscences*," by A. Addison. London: Edward Bull. 1837.—Ed. M. J.]

IX.—ON CERTAIN CASES OF GEOMETRICAL MAXIMA AND MINIMA.

We occasionally meet in Geometry with certain cases of maxima or minima, for which the ordinary analytical process appears to fail, though from geometrical considerations it is obvious that maxima or minima do exist. The explanation of this failure is not given in works on the Differential Calculus, and some notice of it here may be acceptable to our readers. The difficulty and its explanation will be best seen in an example, and none is better suited for the purpose than a question proposed in one of the papers of the Smith's prize examination for 1842. This was—To explain the cause of the failure of the ordinary method of finding maxima and minima, when applied to the problem of finding the greatest or least perpendicular from the focus on the tangent to an ellipse, the perpendicular being expressed in terms of the radius vector.

The usual expression for the perpendicular in terms of the radius vector is

$$p^2 = \frac{b^2 r}{2a - r};$$

and as p^2 will be a maximum or minimum when p is so, the ordinary rule for finding maxima and minima gives us

$$\frac{d}{dr}(p^2) = \frac{2ab^2}{(2a-r)^2} = 0.$$

Now this equation can be satisfied only by $r = \pm\infty$, values which are not admissible in this case; whereas we know from geometry that p is a minimum when $r = a(1 - e)$, and a maximum when $r = a(1 + e)$.

It would appear then that these two values are not given by the analytical process, and the cause of this exception is to be explained. In the general theory of maxima and minima, it is assumed that the independent variable may receive all possible values; whereas in the present case r is limited to those values which are found by assigning all possible values to θ in the expression

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}:$$

in other words, r is not absolutely independent. Now r being a function of another variable, admits itself of maximum and minimum values; and these are the values for which p is a maximum or minimum. The cause of the failure may therefore be thus exhibited: the equation

$$d(p^2) = \frac{2ab^2 dr}{(2a - r)^2} = 0$$

is satisfied by $dr = 0$, that is, by making r a maximum or minimum. Hence generally, if we wish to find the maximum and minimum values of $y = f(x)$, we must consider, not only the values of x which satisfy the equation $\frac{dy}{dx} = 0$, but also the maximum and minimum values of x itself.

In the 7th vol. of *Liouville's Journal*, p. 163, there is given a similar case of failure of the analytical process in the problem—To draw the shortest or longest line to a circle from a point without it. If we take the line passing through the centre of the circle, and the given point O as the axis of x, and call a the distance of the point from the centre, c the radius of the circle, x the co-ordinate of any point P in the circle measured from the centre, we shall have

$$OP^2 = a^2 + c^2 - 2ax, \text{ a max. or min.};$$

from which the usual process would give

$$\frac{d}{dx}(OP^2) = -2a = 0,$$

a nugatory result. In this case x is a maximum or minimum, while OP is a minimum or maximum, and therefore the equation to be satisfied is

$$d\,.\,(OP^2) = -2adx = 0,$$

which is satisfied by $dx = 0$.

In this example the difficulty may be avoided by taking our co-ordinates generally, so that x shall not be a maximum or minimum when OP is so. We shall then have, calling a and b the co-ordinates of the centre of the circle, the other quantities as before,

$$OP^2 = a^2 + b^2 + c^2 - 2b\sqrt{(c^2 - x^2)} - 2ax = \text{max.};$$

whence, by the usual process,

$$\frac{bx}{\sqrt{(c^2 - x^2)}} = a,$$

$$\text{and} \qquad x = \pm \frac{ac}{\sqrt{(a^2 + b^2)}},$$

giving the two values of x, which will solve the problem.

A very instructive example will be found in the problem—To find those conjugate diameters in an ellipse of which the sum is a maximum or a minimum. If r and r_1 be any two conjugate diameters, a, b the axes, we have

$$r + r_1 = \text{maximum or minimum},$$

$$r^2 + r_1^2 = a^2 + b^2 = c^2 \text{ suppose},$$

so that $\quad r + \sqrt{(c^2 - r^2)} = \text{maximum or minimum}.$

From this we have

$$dr\left\{1-\frac{r}{\sqrt{(c^2-r^2)}}\right\}=0.$$

This equation is satisfied either by

$$r=\sqrt{(c^2-r^2)},\quad \text{i.e. by } r=\frac{c}{\sqrt{2}}=r_1,$$

$$\text{or by } dr=0, \text{ which involves } dr_1=0.$$

The former of these results gives the equal conjugate diameters, the sum of which is, as we know, a maximum. The latter result implies that both r and r_1 are maxima or minima, or that they are the principal axes, the sum of which is a minimum. By a different method we might have obtained the minimum instead of the maximum value of $r+r_1$, by the usual process for determining maxima and minima. For since $r^2+r_1^2=a^2+b^2$ and $rr_1 \sin\theta=ab$, θ being the angle between the axes, we have

$$(r+r_1)^2=a^2+b^2+\frac{2ab}{\sin\theta},$$

and hence $$\frac{d}{d\theta}(r+r_1)^2=-\frac{2ab\cos\theta}{(\sin\theta)^2}=0.$$

This is satisfied by $\cos\theta=0$ or $\theta=\frac{1}{2}\pi$, implying that r and r_1 are the principal axes. In this case the maximum value of $r+r_1$ is given by $d\theta=0$, since the equal conjugate diameters are those which make the greatest angle with each other.

G.

X.—ON THE SOLUTION OF CERTAIN FUNCTIONAL EQUATIONS.

By D. F. GREGORY, M.A. Fellow of Trinity College.

In the fourteenth number of this Journal Mr. Leslie Ellis pointed out what appeared to him to be the essential difference between Functional Equations and those which are usually met with in the various branches of analysis. His idea is, that these classes of equations are distinguished by the *order* in which the operations are performed, so that, whereas in our ordinary equations the known operations are performed on those which are unknown, in functional equations the converse is the case, the unknown operations being performed on those which are known. As this view appears to me to be not only correct, but of very great importance for the proper understanding of the higher departments of analysis, I shall endeavour in the following pages to enforce and illustrate it.

On the preceding theory it is easy to see *why* the solution of functional equations must involve difficulties of a higher order than that of equations of the other class. For if we consider an equation as a series of operations performed on a subject, the operations being known and the subject unknown, the solution of the equation involves the finding of the subject, which may be done theoretically by undoing the operations which have been performed on it; that is, by effecting on the second side the inverse of the known operations on the first side. Thus, if we have the equation

$$\frac{dy}{dx} - ay = 0,$$

or, as we may write it,

$$\left(\frac{d}{dx} - a\right) \phi(x) = 0,$$

the object is to find the form of $\phi(x)$, which is readily done by performing the operation $\left(\frac{d}{dx} - a\right)^{-1}$ on both sides, when we have

$$\phi(x) = \left(\frac{d}{dx} - a\right)^{-1} 0 = C\epsilon^{ax}.$$

Here the whole difficulty lies in the performing of the inverse operation; and although practically the difficulty of doing so may be very great, yet it is a difficulty wholly different in *kind* from that which we meet with in trying to solve an equation in which the unknown operation is performed on that which is known. We have then no direct means of disengaging the unknown from the known operations, as the inverse of an unknown operation of course cannot be performed, and the known operation, being the subject, cannot be directly separated from the equation. Thus in the equation

$$\phi(ax) - \phi(x) = 0,$$

where the object is to determine the form of ϕ, we cannot as before write

$$\phi(ax - x) = 0,$$

since the form of ϕ is unknown, and we therefore cannot assume it to be subject to the distributive law; neither can we write

$$a\phi(x) - \phi(x) = 0,$$

since we cannot assume that ϕ and a are commutative operations.

The method which is followed for the solution of certain functional equations, is indicated by the process for the solu-

tion of linear equations in Finite Differences, which are in fact functional equations of a particular form. Thus the equation

$$u_{x+1} - a u_x = 0$$

might be written

$$\phi(x+1) - a\phi(x) = 0,$$

in which the form of ϕ is to be determined.

Here the known operations are the subjects of the unknown, and we cannot directly disengage them; but we are enabled to do so by transforming the equation into one in which the unknown operation ϕ is the subject. For, assuming the operation D to be such that

$$D\phi(x) = \phi(x+1),$$

we are able to investigate the laws of combination of this new symbol and its various properties, so as to make it a known operation. The equation then becomes

$$D\phi(x) - a\phi(x) = 0.$$

Now we can shew that D is a distributive symbol with respect to its subject, and that it is commutative with respect to a; we may therefore write the equation in the form

$$(D - a)\,\phi(x) = 0,$$

whence $$\phi(x) = (D-a)^{-1}\,0.$$

For the complete solution, there remains only that we should know the inverse operations of $D - a$ or D, and these are found from the investigation of its direct action. The result, as we know, is

$$\phi(x) = C a^x.$$

It is useless here to show how this theory may be extended to the solution of general linear equations in finite differences, as that has been sufficiently developed in other places: we shall therefore pass on to show that the same method may be applied to other functional equations.

Let us suppose ω to be any known operation performed on x, so that $\omega(x)$ is a known function of x, and let ϕ be an unknown operation; then the equation

$$\phi(\omega^n x) + a_1\phi(\omega^{n-1}x) + a_2\phi(\omega^{n-2}x) + \&\text{c.} + a_n\phi(x) = X,$$

in which $a_1, a_2, \ldots . a_n$ are constants, and ϕ is a function to be determined, is a functional equation which bears a close analogy to the general linear equation in finite differences, and which may be solved by a similar process.

Let π be the symbol of an operation which, when performed on $\phi(x)$, converts it into $\phi(\omega x)$, so that

$$\pi\phi(x) = \phi(\omega x).$$

This symbol π possesses various properties in common with the symbol D and others, which are often used. Thus, since

$$\pi\pi\phi(x), \text{ or } \pi^2\phi(x) = \pi\phi(\omega x) = \phi(\omega\omega x) = \phi(\omega^2 x),$$

we see generally that when n is an integer

$$\pi^n\phi(x) = \phi(\omega^n x);$$

from which also it is easy to show that

$$\pi^m\pi^n\phi(x) = \pi^{m+n}\phi(x),$$

or the successive operations of π are subject to the index law. Again, we may consider π as a distributive function for

$$\pi\{f(x) + \phi(x)\} = f(\omega x) + \phi(\omega x) = \pi f(x) + \pi\phi(x).$$

Also, since π acts only on a function which involves x, it is commutative with respect to quantities not involving x; so that a being such a quantity,

$$a\pi = \pi a.$$

These are the laws which are used in applying the method of the separation of symbols to the solution of linear differential equations; and hence the same method may be applied to our functional equation. If we introduce into it the symbol π, it becomes

$$\pi^n\phi(x) + a_1\pi^{n-1}\phi(x) + \&c. + a_n\phi(x) = X,$$

which is no longer in a functional form, since the unknown operation ϕ is the subject of known operations. Separating the symbols of operation, we have

$$(\pi^n + a_1\pi^{n-1} + a_2\pi^{n-2} + \&c. + a_n)\phi(x) = X.$$

Now if $r_1, r_2, \ldots . r_n$ be the roots of the equation,

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \&c. + a_n = 0,$$

the complex operation performed on $\phi(x)$ may, in consequence of the laws of combination given above, be decomposed into the simpler operations

$$(\pi - r_1)(\pi - r_2) \ldots . (\pi - r_n)\phi(x) = X,$$

exactly as is done in linear differential equations. And if $N_1, N_2, \ldots . N_n$ be the coefficients of the partial fractions arising from the decomposition of

$$\frac{1}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \&c. + a_n} = \frac{1}{(z - r_1)(z - r_2)\ldots .(z - r_n)},$$

we have, by effecting the inverse operation of that on the left-hand side of the equation,

$$\phi(x) = N_1(\pi - r_1)^{-1}X + N_2(\pi - r_2)^{-1}X + \ldots . + N_n(\pi - r_n)^{-1}X$$
$$+ (\pi - r_1)^{-1}0 + (\pi - r_2)^{-1}0 + \ldots . + (\pi - r_n)^{-1}0.$$

The binomial operations in the first line may be expanded in integral powers of π, that is, according to successive performances of the known operation indicated by ω, and the results may therefore be assumed as known. But the operations in the second line must be developed in negative powers of π, implying the performance of inverse operations; the results of these must of course vary according to the nature of π or ω; and it is plain that any one of them is of the same form as that at which we should arrive in solving the equation

$$(\pi - r)\,\phi(x) = 0, \quad \text{or} \quad \pi\phi(x) - r\phi(x) = 0,$$

which is the simple functional equation

$$\phi(\omega x) - r\phi(x) = 0.$$

This may always be done, or supposed to be done, by Laplace's method, in which it is reduced to the solution of two equations of differences: one of these is always a linear equation of the first order, the other depends on the nature of the function represented by ω.

The preceding analysis shows us, that the solution of a certain class of functional equations may be reduced, exactly like linear equations in differentials and finite differences, to the determination of certain inverse operations, in the performance of which alone the difficulty of the solution lies: one or two examples may be of use in illustrating the theory.

Let $\omega(x) = mx$, m being a constant; and let the equation be of the second degree,

$$\phi(m^2x) + a\phi(mx) + b\phi(x) = x^n.$$

If α, β be the roots of $z^2 + az + b = 0$, this may by the preceding theory be put under the form

$$(\pi - \alpha)(\pi - \beta)\,\phi(x) = x^n,$$

where π is such that $\quad \pi\phi(x) = \phi(mx)$.

Hence $\phi(x) = (\pi - \alpha)^{-1}(\pi - \beta)^{-1}x^n + (\pi - \alpha)^{-1}\,0 + (\pi - \beta)^{-1}\,0.$

The first term of the second side of the equation is easily determined: for since

$$\pi(x^n) = (mx)^n = m^n \,.\, x^n,$$

we may replace π by m^n, so that the term becomes

$$(m^n - \alpha)^{-1}\,(m^n - \beta)^{-1}\,x^n = \frac{x^n}{m^{2n} + am^n + b}.$$

There remains to determine the inverse operations, which are to be found from the solution of the functional equation

$$\phi_1(mx) - \alpha\phi_1(x) = 0 \,\ldots\ldots\ldots (1).$$

Now, by Laplace's method, assume

$$x = u_z, \quad mx = u_{z+1},$$

so that $$u_{z+1} - mu_z = 0 \;\ldots\ldots\ldots\ldots (2),$$

an equation for determining u_z, which, being known, enables us to express z in terms of x. Equation (1) may be written as

$$\phi_1(u_{z+1}) - \alpha\phi_1(u_z) = 0,$$

or simply $$v_{z+1} - \alpha v_z = 0 \;\ldots\ldots (3).$$

The integration of the equations (2) and (3), enables us to solve the given functional equation (3). The solution of (2), on the assumption that the arbitrary function is ~~a constant~~, is

$$u_z = Cm^z = x;$$

whence, by changing the constant, we have

$$z = \frac{\log(cx)}{\log m}.$$

In like manner the solution of (3) is

$$b_z = C\alpha^z = C\varepsilon^{\frac{\log \alpha}{\log m}\log(cx)} = C(cx)^{\frac{\log\alpha}{\log m}} = \phi_1(x),$$

C being an arbitrary function of $\sin 2\pi z$ and $\cos 2\pi z$.

Similarly for β: hence the solution of the given functional equation is

$$\phi(x) = \frac{x^n}{m^{2n} + am^n + b} + C(cx)^{\frac{\log\alpha}{\log m}} + C'(cx)^{\frac{\log\beta}{\log m}}.$$

Again, let $\omega(x) = x^n$, and the functional equation be

$$\phi(x^{n^2}) + a\phi(x^n) + b\phi(x) = \log x.$$

If we assume $x = u_z$, $x^n = u_{z+1}$, the solution of the preceding equation will depend on that of

$$u_{z+1} = u_z^{\,n}, \text{ and of } v_{z+1} - \alpha v_z = 0.$$

The integral of the former is

$$u_z = c^{n^z} = x;$$

whence, by a change of constant,

$$z = \frac{1}{\log n}\log\log(x^c).$$

The integral of the latter is

$$v_z = C\alpha^z = C\varepsilon^{\frac{\log\alpha}{\log n}\log(\log x^c)} = C(\log x^c)^{\frac{\log\alpha}{\log n}}.$$

Also, if $\phi(x^n) = \pi\phi(x)$, we have

$$\pi\log(x) = \log(x^n) = n\log x,$$

and therefore

$$(\pi^2 + a\pi + b)^{-1} \log x = \frac{\log x}{n^2 + an + b}.$$

Hence the solution of the proposed equation is

$$\phi(x) = \frac{\log x}{n^2 + an + b} + C(\log x^c)^{\frac{\log \alpha}{\log n}} + C_1(\log x^c)^{\frac{\log \beta}{\log n}}.$$

If the function $\omega(x)$ be a periodic function of the n^{th} order, so that $\omega^n(x) = x$, $\omega^{n+1}(x) = \omega(x)$, &c., the result of such an operation as

$$(\pi - r)^{-1} f(x),$$

can be always readily determined. For, if we expand the binomial in ascending powers of π, it becomes

$$-\frac{1}{r}\left(1 + \frac{\pi}{r} + \frac{\pi^2}{r^2} + \text{\&c.} + \frac{\pi^{n-1}}{r^{n-1}} + \frac{\pi^n}{r^n} + \text{\&c.} + \frac{\pi^{2n}}{r^{2n}} + \text{\&c.}\right) f(x).$$

But as $\pi^n f(x) = f(\omega^n x) = f(x)$, this is equivalent to

$$-\frac{1}{r}\left(1 + \frac{1}{r^n} + \frac{1}{r^{2n}} + \text{\&c.}\right)\left(1 + \frac{\pi}{r} + \frac{\pi^2}{r^2} + \text{\&c.} + \frac{\pi^{n-1}}{r^{n-1}}\right) f(x)$$

$$= \frac{1}{1 - r^n}\{\pi^{n-1} + r\pi^{n-2} + \text{\&c.} + r^{n-2}\pi + r^{n-1}\} f(x)$$

$$= \frac{1}{1 - r^n}\{f(\omega^{n-1}x) + rf(\omega^{n-2}x) + \text{\&c.} + r^{n-2}f(\omega x) + r^{n-1}f(x)\}.$$

As an example, let us assume

$$\omega(x) = \frac{1 + x}{1 - x},$$

which is a periodic function of the fourth order, the successive results being

$$\omega^2(x) = -\frac{1}{x}, \quad \omega^3(x) = \frac{x - 1}{x + 1}, \quad \omega^4(x) = x.$$

Let the functional equation be

$$\phi\left(\frac{1 + x}{1 - x}\right) - a\phi(x) = x.$$

Then if $\quad x = u_z, \quad \dfrac{1 + x}{1 - x} = u_{z+1},$

$$u_{z+1}u_z - u_{z+1} + u_z + 1 = 0.$$

The solution of this is (Herschel's *Examples*, p. 34,)

$$u_z = \tan\left(C + \frac{\pi}{4} z\right) = x;$$

whence $\quad z = \dfrac{4}{\pi}(\tan^{-1} x - C).$

The solution of the equation

$$\phi\left(\frac{1+x}{1-x}\right) - a\phi(x) = 0,$$

is therefore

$$\phi(x) = Ca^z = Ca^{\frac{4}{\pi}(\tan^{-1} x - C)} = C_1 a^{\frac{4}{\pi}\tan^{-1} x},$$

by changing the arbitrary constant. Hence the proposed functional equation gives

$$\phi(x) = (\pi - a)^{-1}\, x + C_1 a^{\frac{4}{\pi}\tan^{-1} x}$$

$$= \frac{1}{1-a^4}\left(\frac{x-1}{x+1} - \frac{a}{x} + a^2\,\frac{1-x}{1+x} + a^3 x\right) + C_1 a^{\frac{4}{\pi}\tan^{-1} x}.$$

Again, let $\omega(x) = \dfrac{a^2}{x}$, a periodic function of the second order, and let the functional equation be

$$\phi\left(\frac{a^2}{x}\right) + \phi(x) = \varepsilon^{nx}.$$

Then if $x = u_z$, $\dfrac{a^2}{x} = u_{z+1}$, we have

$$u_{z+1} u_z = a^2;$$

the integral of which is $u_z = aC^{(-1)^z} = x$;

$$\text{whence} \quad (-1)^z = \left(\log \frac{x}{a}\right)^c.$$

But the functional equation gives us

$$v_{z+1} + v_z = 0,$$

$$\text{whence} \quad v_z = C(-1)^z = C\left(\log\frac{x}{a}\right)^c.$$

$$\text{Hence} \quad \phi(x) = (\pi + a)^{-1}\,\varepsilon^{nx} + C\left(\log\frac{x}{a}\right)^c$$

$$= \frac{1}{1-a^2}\left(\varepsilon^{nx} - a\,\varepsilon^{\frac{na^2}{x}}\right) + C\left(\log\frac{x}{a}\right)^c.$$

In conclusion, I may observe that this article does not pretend to give any new results, as the solution of Functional Equations of the kind here treated of is already known, (see Herschel's *Finite Differences*, p. 547): the object of it is merely to illustrate the theory before spoken of, and to show that a method which has been found useful in two departments of analysis may likewise be applied to simplify the processes of a more difficult branch.

XI.—MATHEMATICAL NOTES.

1. *Note on the Theory of Numbers.*—The following elegant demonstration of a known remarkable proposition in the Theory of Numbers, is given by M. E. Catalan in *Liouville's Journal*, tom. IV. p. 7.

If $\phi(n)$ represent the number of integers which are less than n and prime to it, and if d, d', d'', &c. be the divisors of a number N,

$$\phi(d) + \phi(d') + \phi(d'') + \text{\&c.} = N.$$

Let $N = a^\alpha b^\beta c^\gamma \ldots$, a, b, c, being the prime factors of N; then any divisor d may be represented by $a^i b^k c^l \ldots$, where the exponents $i, k, l \ldots$ vary from 0 to α, 0 to β, 0 to γ, &c. By a known theorem,

$$\phi(d) = d \cdot \frac{a-1}{a} \cdot \frac{b-1}{b} \cdot \frac{c-1}{c} \ldots$$
$$= a^{i-1}(a-1) \cdot b^{k-1}(b-1) \cdot c^{l-1}(c-1) \ldots,$$

each of the factors of the latter product being replaced by unity when the exponent involved in it is equal to -1. Now all the values of this product due to the different values which can be assigned to the exponents are given in the terms of the product of the quantities

$$1 + (a-1) + a(a-1) + a^2(a-1) + \ldots + a^{\alpha-1}(a-1),$$
$$1 + (b-1) + b(b-1) + b^2(b-1) + \ldots + b^{\beta-1}(b-1),$$
$$1 + (c-1) + c(c-1) + c^2(c-1) + \ldots + c^{\gamma-1}(c-1),$$
&c. &c.

Therefore, cancelling the terms which destroy each other,

$$\phi(d) + \phi(d') + \phi(d'') + \text{\&c.} = a^\alpha b^\beta c^\gamma \ldots = N.$$

2. *To shew that if*

$$l^2 + m^2 + n^2 = 1 \ldots\ldots (1),$$
$$l'^2 + m'^2 + n'^2 = 1 \ldots\ldots (2),$$
$$l''^2 + m''^2 + n''^2 = 1 \ldots\ldots (3),$$
$$l'l'' + m'm'' + n'n'' = 0 \ldots\ldots (4),$$
$$l''l + m''m + n''n = 0 \ldots\ldots (5),$$
$$ll' + mm' + nn' = 0 \ldots\ldots (6).$$

Then shall

$$l^2 + l'^2 + l''^2 = 1 \ldots\ldots (7),$$
$$m^2 + m'^2 + m''^2 = 1 \ldots\ldots (8),$$
$$n^2 + n'^2 + n''^2 = 1 \ldots\ldots (9),$$
$$mn + m'n' + m''n'' = 0 \ldots\ldots (10),$$
$$nl + n'l' + n''l'' = 0 \ldots\ldots (11),$$
$$lm + l'm' + l''m'' = 0 \ldots\ldots (12).$$

Putting (1) under the form

$$ll + mm + nn = 1, \text{ multiply by } m''n' - n''m',$$

$$(5) \ldots\ldots l''l + m''m + n''n = 0, \quad \ldots\ldots \quad m'n - n'm,$$

$$(6) \ldots\ldots l'l + m'm + n'n = 0, \quad \ldots\ldots \quad mn'' - nm''.$$

Then adding, we have, as in ordinary cross multiplication,

$$\{l(m''n' - n''m') + l''(m'n - n'm'') + l'(mn'' - nm'')\}\, l = m''n' - n''m',$$

or, for brevity, $\qquad Sl = m''n' - n''m'.$

Hence, on account of the symmetry of S,

$$S = \frac{m''n' - n''m'}{l} = \frac{n''m - nm''}{l'} = \frac{nm' - n'm}{l''} \ldots\ldots (a),$$

and hence

$$l^2S + l'^2S + l''^2S = l(m''n' - m'n'') + l'(n''m - m''n) + l''(nm' - n'm),$$

$$= S,$$

$$\text{therefore} \quad l^2 + l'^2 + l''^2 = 1.$$

Similarly (8) and (9) may be proved. Again, from (a),

$$lmS + l'm'S + l''m''S = 0,$$

$$\text{therefore} \quad lm + l'm' + l''m'' = 0.$$

And in a similar manner (10) and (11) may be proved.

We have also, from (a),

$$(l^2 + l'^2 + l''^2)\, S^2 = (m''n' - n''m')^2 + (n''m - m''n)^2 + (nm' - n'm)^2$$

$$= (m^2 + m'^2 + m''^2)\,(n^2 + n'^2 + n''^2), \text{ on account of } (10),$$

$$= 1,$$

$$\text{therefore} \quad S^2 = 1.$$

Hence by (a) we can find l, l', l'', in terms of the other six quantities.

T.

CORRIGENDA.

Vol. III. p. 165, Equation (37), *for* L *read* $\mathfrak{L}$.

.. p. 166, Equation (38), *for* $L - \frac{1}{kr}$, $L' - \frac{1}{kr'}$, *read* $\mathfrak{L} - \frac{1}{kr}$, $\mathfrak{L}' - \frac{1}{kr'}$.

.. p. 164-166, passim, *for* $\theta - \pi$, $\theta' - \pi'$, $\pi - \pi'$, *read* $\theta - \omega$, $\theta' - \omega'$, $\omega - \omega'$.